

AD-A054 882

DELAWARE UNIV NEWARK DEPT OF STATISTICS AND COMPUTER--ETC F/G 12/1
SOME EXPLICIT FORMULAS AND COMPUTATIONAL METHODS FOR INFINITE S--ETC(U)
APR 78 V RAMASWAMI, M F NEUTS

AFOSR-77-3236

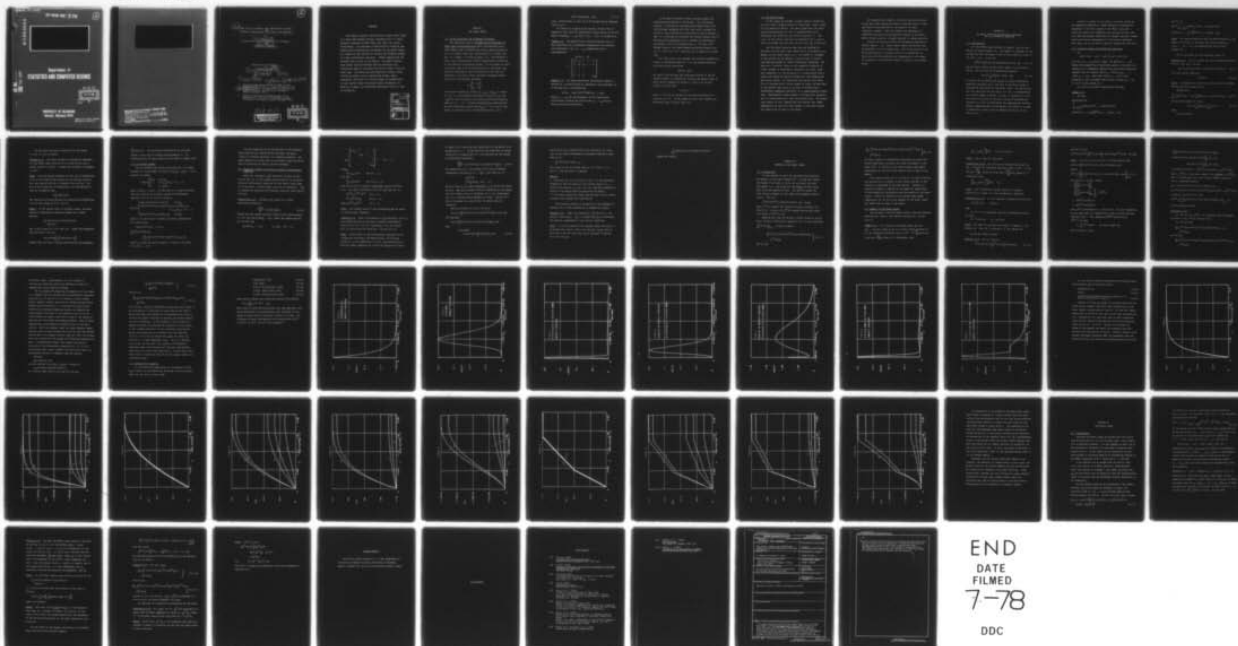
UNCLASSIFIED

TR-78/5

AFOSR-TR-78-0981

NL

| OF |
AD
A054 882

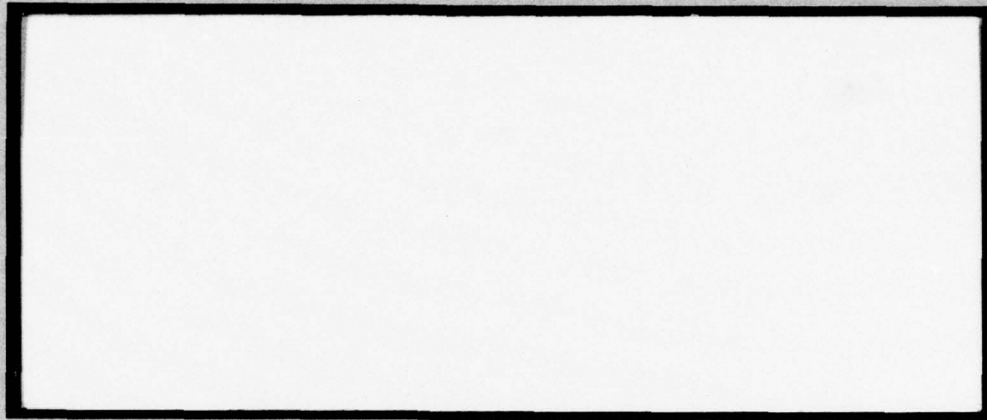


END
DATE
FILMED
7-78
DDC

FOR FURTHER TRAN

(2)

AD A 054882



Department of
STATISTICS AND COMPUTER SCIENCE

AD No. _____
DDC FILE COPY

DDC
RECEIVED
JUN 12 1978
B

UNIVERSITY OF DELAWARE
Newark, Delaware 19711

Approved for public release;
distribution unlimited.

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFOSR)
NOTICE OF TRANSMITTAL TO DDC
This technical report has been reviewed and is
approved for public release IAW AFR 190-18 (Vb).
Distribution is unlimited.
A. D. BLOSE
Technical Information Officer

2

6

SOME EXPLICIT FORMULAS AND COMPUTATIONAL METHODS
FOR
INFINITE SERVER QUEUES WITH PHASE TYPE ARRIVALS.

by

10

V. Ramaswami

Purdue University and University of Delaware

and

Marcel F. Neuts

University of Delaware

Department of Statistics
and

9

Computer Science

Technical Report, No. 78/5

14

TR-

11

April 1978

12

65p.

16

2305

17

A5

18

AFOSR

19

TR-78-0981

1. Research supported by AFOSR-72-2350C at the
Department of Statistics, Purdue University and
15 AFOSR-77-3236 at the Department of Statistics &
Computer Science, University of Delaware,
Newark, Delaware 19711.

2. Research supported by AFOSR-77-3236 at the
Department of Statistics & Computer Science,
University of Delaware, Newark, Delaware 19711.

1473

391 807

DISTRIBUTION STATEMENT A

Approved for public release;
Distribution Unlimited

DDC

RECEIVED
JUN 12 1978
REGULATED

B

4B

ABSTRACT

This paper discusses infinite server queues whose input is a Phase Type Renewal Process introduced by M. F. Neuts (Renewal Processes of Phase Type, Nav. Res. Log. Quart., forthcoming). The problems of obtaining the transient and steady-state distributions and moments of the queue length are reduced to the solution of certain well-behaved systems of linear differential equations. Sample computations are provided with as many as ten phases. The paper contains some useful explicit formulas and also discusses the interesting special case where the service time is also of phase type. The Phase Type Distributions include a wide variety of models such as generalized Erlang, hyper-exponential (mixtures of a finite number of exponentials) as very special cases and possess great versatility in modeling a number of interesting qualitative features such as bimodality.

ACCESSION for	
NTIS	Write Section <input checked="" type="checkbox"/>
DDC	Edw. Section <input type="checkbox"/>
UNANNOUNCED	<input type="checkbox"/>
JUSTIFICATION	
BY	
DISTRIBUTION/AVAILABILITY CODES	
Dist.	AVAIL. and/or SPECIAL
A	

CHAPTER I

THE PH/G/ ∞ QUEUE

1.1 PH-Distributions and PH-Renewal Processes

The versatile class of probability distributions of Phase Type (PH-distributions) which includes many well-known models such as generalized Erlang, hyper-exponential etc., as special cases was introduced by M. F. Neuts [5], who, in a number of follow-up papers (c.f. Bibliography in [7]), also discussed their useful computational properties in the analysis of queueing systems. In this paper we shall exclusively deal with PH-distributions of continuous type which are obtained as the distribution of the time till absorption in an $(m+1)$ -state continuous parameter Markov chain with infinitesimal generator

$$Q = \begin{bmatrix} T & \underline{I}^0 \\ \underline{0} & 0 \end{bmatrix}$$

and initial probability vector $(\underline{\alpha}, \alpha_{m+1})$, $0 \leq \alpha_{m+1} < 1$, where $T = (T_{ij})$ is a non-singular $m \times m$ matrix such that $T_{ii} < 0$ and $T_{ij} \geq 0$ for $i \neq j$, and $\underline{I}^0 > \underline{0}$ is an m -vector satisfying $T\underline{e} + \underline{I}^0 = \underline{0}$, where $\underline{e}' = (1, \dots, 1)$. To avoid uninteresting complications, we shall henceforth assume that $\alpha_{m+1} = 0$. It is now an easy matter to see that such a PH-distribution has c.d.f.,

$$F(x) = 1 - \underline{\alpha} \exp(Tx) \underline{e}, \quad x \geq 0. \quad (1.1.1)$$

Such a distribution is said to be of PH-type with a representation $(\underline{\alpha}, T)$.

For technical reasons we may assume, without loss of generality [5], that the conservative stable matrix $Q^* = T + T^0 A^0$, where $A^0 = \text{diag}(\alpha_1, \dots, \alpha_m)$ and $T^0 = (\underline{1}^0, \dots, \underline{1}^0)$, is irreducible.

Example 1.1: The generalized Erlang distribution which is the convolution of m independent exponential distributions with parameters, say, μ_1, \dots, μ_m respectively has a representation

$$\underline{\alpha} = (1, 0, \dots, 0)$$

$$T = \begin{bmatrix} -\mu_1 & \mu_1 & 0 & \dots & 0 \\ 0 & -\mu_2 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\mu_m \end{bmatrix}$$

Example 1.2: The hyper-exponential distribution (which is defined as a finite mixture of exponential distributions) is of PH-type with a representation

$$\underline{\alpha} = (\alpha_1, \dots, \alpha_m), \text{ and } T = \text{diag}(-\mu_1, \dots, -\mu_m),$$

where μ_1, \dots, μ_m are the parameters of the exponential distributions forming the mixture and $\alpha_1, \dots, \alpha_m$ are the ratios governing the mixture.

In [6] Neuts discussed renewal processes where the underlying distribution is of PH-type. The "PH-Renewal Process" is obtained by considering the Markov Chain with infinitesimal generator $Q^* = T + T^0 A^0$ and initial probability vector $\underline{\alpha}$, and a constructive definition of such a process is that it is obtained by resetting the original Markov Chain Q following each absorption (i.e., renewal) by performing a multinomial trial with probabilities $\underline{\alpha}$. For such a PH-Renewal Process, the times between successive renewals are i.i.d., with c.d.f. $F(\cdot)$ given by (1.1.1) which is of Phase Type.

For later use we let $\underline{\theta}$ denote the invariant probability vector of the Markov Chain Q^* , i.e., the unique (strictly positive) vector satisfying

$$\underline{\theta} Q^* = \underline{0}, \quad \underline{\theta} \underline{e} = 1.$$

We recall from [6] that the stationary version of the PH-Renewal Process is obtained by starting the Markov Chain Q^* with initial probability vector $\underline{\theta}$. Also it is an easy matter to verify that

$$\underline{\theta} = -\lambda \underline{\alpha} T^{-1},$$

where $\lambda^{-1} = -\underline{\alpha} T^{-1} \underline{e}$ is the mean of the PH-distribution $F(\cdot)$ given by (1.1.1). In the sequel we shall let θ denote the $m \times m$ matrix each of whose rows is $\underline{\theta}$.

1.2 The PH/G/ ∞ Queue

In this paper we consider a GI/G/ ∞ queue in which the arrivals form a renewal process of Phase Type. Such a model will be denoted by PH/G/ ∞ . We shall assume that the underlying PH-distribution $F(\cdot)$ has a representation $(\underline{\alpha}, T)$ satisfying the conditions set forth in Section 1.1. The c.d.f. of the service time will be denoted by $\tilde{H}(\cdot)$, and it shall be assumed that the mean service time $\mu = \int_0^{\infty} \{1 - \tilde{H}(t)\} dt < \infty$.

For the PH/G/ ∞ queue we show that the problems of obtaining the transient (and steady-state) distributions and moments of the queue length (i.e., the number of customers in the system) can be reduced to the solution of certain well-behaved systems of linear differential equations. By applying a classical result on the asymptotic behavior of linear systems of differential equations, we prove, under the assumption $\mu < \infty$, the existence of a steady-state distribution and obtain an explicit formula for the steady-state mean queue length. By considering the special case where $\tilde{H}(\cdot)$ is discrete with a finite number of atoms, we show that in the general case there is no hope of solving these differential equations explicitly in a computationally useful form. Nevertheless, these systems of differential equations can be solved numerically with considerable ease. We discuss some aspects of such computations and present some sample computations of the first four moments of the queue length for models with as many as ten phases.

Our computational examples illustrate how sole reliance on the mean queue length may result in serious errors in the qualitative interpretation of the behavior of these stochastic systems. They also enhance the importance of algorithmic solutions using numerical methods to stochastic models without which an in-depth analysis is seldom possible.

Finally, we discuss briefly the interesting subclass of PH/PH/ ∞ queues - i.e., PH/G/ ∞ queues where the service time distribution is also of phase type - and in the course of the discussion obtain an alternate proof of a theorem due to Neuts and Jain [8] governing the independence of the number of customers in the different phases of service in an M/PH/ ∞ queue.

CHAPTER II

THE BASIC SYSTEM OF DIFFERENTIAL EQUATIONS AND ITS ASYMPTOTIC BEHAVIOR

2.1 Introduction

For the PH/G/ ∞ queue defined in Chapter I we let $X(t)$ denote the queue length (i.e., the number of customers in the system) and $J(t)$ the phase of the arrival process at time $t+$. For $k \geq 0$, $i, j = 1, \dots, m$ we let

$$G_k^{ij}(t) = P[X(t) = k, J(t) = j | X(0) = 0, J(0) = i], \quad t \geq 0 \quad (2.1.1)$$

and let $G_k(t)$ denote the $m \times m$ matrix whose (i, j) -th entry is $G_k^{ij}(t)$. We also define the generating function

$$G(z, t) = \sum_{k=0}^{\infty} z^k G_k(t), \quad |z| \leq 1, \quad t \geq 0. \quad (2.1.2)$$

In section 2.2 we derive the system of linear differential equations governing $G(z, \cdot)$ and call it the basic system of differential equations for the PH/G/ ∞ queue. By considering the case where the service time c.d.f. $H(\cdot)$ is concentrated on a finite number of points, we show, in Section 2.3, that the explicit solution to the basic system of differential equations is of no use whatsoever for computational purposes thereby suggesting that in the general case there is no hope of obtaining a useful explicit formula for $G(z, t)$ or its limit as $t \rightarrow \infty$.

Finally in Section 2.4 we apply a classical result on the asymptotic behavior of linear systems of differential equations to the basic system of the PH/G/ ∞ queue and establish (under our assumption that the mean service time $\mu < \infty$) that the process $(X(t), J(t))$ is ergodic and has a steady state distribution independent of the initial conditions. Once again, we do not have an explicit formula for the limit.

2.2 The Basic System of Differential Equations

Let

$$S_n(t) = \{(u_1, \dots, u_n) : u_i \geq 0, \sum_{i=1}^n u_i \leq t\}, t \geq 0, n \geq 1.$$

Also for $(u_1, \dots, u_n) \in S_n(t)$, $0 \leq k \leq n$, let $g_n^{(k)}(t; u_1, \dots, u_n)$ denote the probability of obtaining exactly k successes in a sequence of n independent Bernoullian trials with respective probabilities of success $1 - \tilde{H}(t - u_1)$, $1 - \tilde{H}(t - u_1 - u_2)$, \dots , $1 - \tilde{H}(t - u_1 - \dots - u_n)$. Note that $1 - \tilde{H}(t - u_1 - \dots - u_i)$ is the probability that a customer who arrives at epoch $u_1 + \dots + u_i$ is still in the system at t .

In terms of the above notations we now have

Lemma 2.2.1:

For $t \geq 0$,

$$G_0(t) = \exp(Tt) +$$

$$\sum_{n=1}^{\infty} \int_{S_n(t)} \exp(Tu_1) \cdot T^\circ A^\circ \cdot \dots \cdot \exp(Tu_n) \cdot T^\circ A^\circ \cdot$$

$$\exp\{T(t - u_1 - \dots - u_n)\} g_n^{(0)}(t; u_1, \dots, u_n) du_1 \dots du_n$$

and for $k \geq 1$

$$G_k(t) = \sum_{n=k}^{\infty} \int_{S_n(t)} \exp(Tu_1) \cdot T^\circ A^\circ \cdot \dots \cdot \exp(Tu_n) \cdot T^\circ A^\circ \cdot \exp\{T(t-u_1-\dots-u_n)\} g_n^{(k)}(t; u_1, \dots, u_n) du_1 \dots du_n$$

Proof: The formulae above are got by conditioning on n , the number of arrivals in $(0, t]$, and the arrival epochs $u_1, u_1+u_2, \dots, u_1+\dots+u_n$, and applying the law of total probability.

We are now ready to state our basic result as

Theorem 2.2.2: For all $t \in C(\tilde{H})$, the set of continuity points of $\tilde{H}(\cdot)$, and $|z| \leq 1$, we have

$$\frac{\partial}{\partial t} G(z, t) = [(T + T^\circ A^\circ) + (z-1)\{1 - \tilde{H}(t)\} T^\circ A^\circ] G(z, t) \quad (2.2.3)$$

with the initial condition

$$G(z, 0) = I \quad (2.2.4)$$

Proof: (Throughout this proof we shall use prime to denote derivatives with respect to t).

Noting that

$$\sum_{k=0}^n z^k g_n^{(k)}(t; u_1, \dots, u_n) = \prod_{i=1}^n [\tilde{H}(t-u_1-\dots-u_i) + z\{1 - \tilde{H}(t-u_1-\dots-u_i)\}],$$

we have from Lemma 2.2.1 that

$$G(z, t) = \sum_{k=0}^{\infty} z^k G_k(t) = \sum_{n=0}^{\infty} C_n(z, t), \quad (2.2.5)$$

where

$$C_0(z, t) = \exp(Tt)$$

and for $n \geq 1$

$$C_n(z, t) =$$

$$S_n(t) \left\{ \prod_{i=1}^n \exp(Tu_i) \cdot T^\circ A^\circ \cdot [\tilde{H}(t-u_1-\dots-u_i) + z\{1-\tilde{H}(t-u_1-\dots-u_i)\}] \right\} \cdot \exp[T(t-u_1-\dots-u_n)] du_1 \dots du_n$$

Now for $n \geq 2$,

$$\begin{aligned} C_n(z, t) &= \int_0^t \exp(Tu_1) \cdot T^\circ A^\circ \cdot [\tilde{H}(t-u_1) + z\{1-\tilde{H}(t-u_1)\}] \cdot \\ &\quad S_{n-1}(t-u_1) \left\{ \prod_{i=2}^n \exp(Tu_i) \cdot T^\circ A^\circ \cdot [\tilde{H}(t-u_1-\dots-u_i) + \right. \\ &\quad \left. z\{1-\tilde{H}(t-u_1-\dots-u_i)\}] \right\} \cdot \exp[T(t-u_1-\dots-u_n)] \cdot \\ &\quad du_2 \dots du_n \Big] du_1 \\ &= \int_0^t \exp(Tu_1) \cdot T^\circ A^\circ [\tilde{H}(t-u_1) + z\{1-\tilde{H}(t-u_1)\}] C_{n-1}(z, t-u_1) du_1 \\ &= \int_0^t \exp[T(t-\tau)] \cdot T^\circ A^\circ [\tilde{H}(\tau) + z\{1-\tilde{H}(\tau)\}] C_{n-1}(z, \tau) d\tau \end{aligned}$$

or

$$\exp(-Tt) \cdot C_n(z, t) = \int_0^t \exp(-T\tau) \cdot T^\circ A^\circ [z + (1-z)\tilde{H}(\tau)] C_{n-1}(z, \tau) d\tau.$$

This, on differentiating with respect to t , yields

$$-TC_n(z, t) + C'_n(z, t) = T^\circ A^\circ [z + (1-z)\tilde{H}(t)] C_{n-1}(z, t) \quad (2.2.6)$$

for all $n \geq 2$ and $t \in C(\tilde{H})$. It is easily verified that (2.2.6) holds for $n=1$ also. Now adding (2.2.6) for $n \geq 1$ and using (2.2.5) we get

$$-T[G(z, t) - C_0(z, t)] + [G'(z, t) - C'_0(z, t)] = T^\circ A^\circ [z + (1-z)\tilde{H}(t)] G(z, t)$$

and since

$$C'_0(z,t) = -TC_0(z,t)$$

we have

$$G'(z,t) = [(T + T^0 A^0) + (z-1)\{1 - \tilde{H}(t)\}T^0 A^0]G(z,t)$$

for all $t \in C(\tilde{H})$. Thus we have (2.2.3). Equation (2.2.4) is obvious.

Remark: The system of differential equations (2.2.3) with the initial condition (2.2.4) shall be called the Basic System of Differential Equations for the PH/G/ ∞ queue.

Note that the basic system of differential equations for the PH/G/ ∞ queue is of the form

$$Y'(t) = [A + R(t)]Y(t), \quad (2.2.7)$$

where A is a constant matrix and $R(t) \rightarrow 0$ as $t \rightarrow \infty$. Regarding the system (2.2.7) we have the following

Proposition 2.2.8:

a) If $R(\cdot)$ is Riemann-integrable in $[0, t_0]$, then there exists a unique solution of (2.2.7) in $[0, t_0]$ for any given initial condition.

b) If $\int_0^\infty \|R(\tau)\| d\tau < \infty$ and if all the solutions of $Y'(t) = AY(t)$ are bounded, then all the solutions of (2.2.7) are bounded in $[0, \infty)$.

Proof: For Part (a) we refer the reader to Bellman [1], page 165. Part (b) due to Dini-Hukuhara may be found in Cesari [2], p. 37, 3.3 (iii).

We can apply the above Proposition to the system (2.2.3), (2.2.4) to obtain

Theorem 2.2.9: The basic system of differential equations for the PH/G/ ∞ queue given by (2.2.3) and (2.2.4) has a unique solution in $[0, \infty)$. Further this solution is bounded in $[0, \infty)$.

Proof: That the second condition of Part (b) of Proposition 2.2.8 is satisfied by the system (2.2.3) follows from the fact that $\exp[(T+T^0A^0)t]$ is stochastic for any $t \geq 0$. The rest of the conditions in Proposition 2.2.8 follow easily from our assumption that

$$\mu = \int_0^{\infty} \{1 - \tilde{H}(t)\} dt < \infty.$$

The Theorem now follows directly by specializing Proposition 2.2.8 to the system (2.2.3), (2.2.4).

Remark: In the special case of the M/G/ ∞ queue, the basic system of differential equations reduces to a single equation

$$\begin{aligned} \frac{\partial}{\partial t} G(z, t) &= (z-1) \{1 - \tilde{H}(t)\} \lambda G(z, t) \\ G(z, 0) &= 1, \end{aligned}$$

for, in this case $T = -\lambda$, $T^0 = \lambda$ and $\alpha_1 = 1$. Under the assumption $\mu = \int_0^{\infty} \{1 - \tilde{H}(t)\} dt < \infty$, we have

$$G(z, t) = \exp \left\{ -\lambda \int_0^t \{1 - \tilde{H}(\tau)\} d\tau (1-z) \right\}$$

showing that $X(t)$ has a Poisson distribution with parameter

$\lambda \int_0^t \{1 - \tilde{H}(\tau)\} d\tau$. The stationary distribution of the queue length in this case is Poisson with parameter $\lambda\mu$. An alternate proof of these results can be found in Takács [10].

2.3 A Discrete Example

We now consider the special case where $\tilde{H}(\cdot)$ is concentrated on a finite number of points $0 < t_0 \leq t_1 \leq \dots \leq t_k < \infty$. To be specific we assume

$$1 - \tilde{H}(t) = \begin{cases} 1 = \beta_0 & \text{if } t < t_0 \\ \beta_i & \text{if } t_{i-1} \leq t < t_i; i=1, \dots, k \\ 0 & \text{if } t_k \leq t < \infty \end{cases}$$

where $1 = \beta_0 > \beta_1 > \dots > \beta_k > 0$. In this case it is easily verified that the solution to the basic system of differential equations (2.2.3), (2.2.4) is given by

$$G(z, t) = \begin{cases} \exp\{[(T + T^\circ A^\circ) + (z-1)T^\circ A^\circ]t\} & \text{if } t \leq t_0 \\ \exp\{[(T + T^\circ A^\circ) + (z-1)\beta_i T^\circ A^\circ](t - t_{i-1})\} \cdot G(z, t_{i-1}) & \text{if } t_{i-1} \leq t \leq t_i, i=1, \dots, k \\ \exp[(T + T^\circ A^\circ)(t - t_k)] \cdot G(z, t_k) & \text{if } t \geq t_k \end{cases}$$

which is in the form of a product of matrix exponentials.

Also noting that

$$\exp\{(T + T^\circ A^\circ)t\} \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

$$\lim_{t \rightarrow \infty} G(z, t) = 0 G(z, t_k)$$

$$= 0 \prod_{j=k}^0 \exp\{[(T + T^\circ A^\circ) + (z-1)\beta_j T^\circ A^\circ](t_j - t_{j-1})\}$$

where $t_{-1} = 0$ and the matrix product is taken in the order $j=k, j=k-1, \dots, j=0$.

The non-commutativity of the matrices in the products above prevents any simplification and makes the above "explicit" formulae worthless for computing purposes. The above example also shows that in the general case one has no hope of obtaining any useful explicit formulae.

2.4 Asymptotic Nature of the Basic System of Differential Equations

Under the assumption $\mu = \int_0^{\infty} \{1 - \tilde{H}(t)\} dt < \infty$ we show in this section that as $t \rightarrow \infty$ the unique solution $G(z, t)$ to the basic system of differential equations for the PH/G/ ∞ queue tends to a limit $G(z)$, a matrix whose rows are all identical. This is obtained by applying the following classical result due to Levinson.

Proposition 2.4.1: Consider the system of m linear differential equations

$$\frac{dy(t)}{dt} = [A + R(t)]y(t). \quad (2.4.2)$$

Assume that the Jordan Canonical form of the constant matrix A is of the form $\text{diag}(B_0, \dots, B_s)$, where the square matrices B_j are such that

$$B_0 = \text{diag}(\mu_1, \dots, \mu_\ell); \quad \text{Re } \mu_j \geq 0, \quad j = 1, \dots, \ell,$$

$$B_j = \begin{bmatrix} \mu_{\ell+j} & 1 & & \\ & \mu_{\ell+j} & 1 & \\ & & \ddots & \ddots \\ & & & \mu_{\ell+j} \end{bmatrix}; \quad j=1, \dots, s.$$

Assume

$$\operatorname{Re} \mu_{\ell+j} < -\beta < 0 \quad \text{for } j=1, \dots, s$$

and that

$$\int_0^{\infty} |r_{ij}(t)| dt < \infty \quad \text{for } i, j=1, \dots, m.$$

Then there exist m linearly independent vectors $\underline{y}^{(k)}(t)$; $k=1, \dots, m$, each a solution of (2.4.2) such that as $t \rightarrow \infty$,

$$\underline{y}^{(k)}(t) \sim e^{\mu_k t} \underline{c}^{(k)}; \quad k=1, \dots, \ell$$

$$e^{\beta t} \underline{y}^{(k)}(t) \rightarrow \underline{0}; \quad k=\ell+1, \dots, m,$$

$$\text{where } A \underline{c}^{(k)} = \mu_k \underline{c}^{(k)}; \quad k=1, \dots, \ell.$$

Proof: The lengthy proof of this Proposition may be found in Levinson [3], Theorem 3.

Theorem 2.4.3: Under the assumption $\mu = \int_0^{\infty} \{1 - \tilde{H}(t)\} dt < \infty$, $G(z, t)$, the unique solution to the basic system of differential equations (2.2.3), (2.2.4), converges as $t \rightarrow \infty$ to a matrix $G(z)$ all whose rows are identical. Further $G(1-) = 0$.

Proof: Since $T + T^{\circ} A^{\circ}$ is the infinitesimal generator of an irreducible continuous time Markov-Chain, $(T + T^{\circ} A^{\circ}) \underline{e} = \underline{0}$. Further 0 , as an eigenvalue of $T + T^{\circ} A^{\circ}$, has multiplicity 1. Also any other eigenvalue of $T + T^{\circ} A^{\circ}$ has negative real part.

In short $T+T^{\circ}A^{\circ}$ satisfies the conditions for the matrix A in Proposition 2.4.1. It may easily be verified that all other conditions of Proposition 2.4.1 are satisfied by the system of differential equations

$$\frac{\partial \underline{g}(t)}{\partial t} = [(T+T^{\circ}A^{\circ})+(z-1)\{1-\tilde{H}(t)\}T^{\circ}A^{\circ}]\underline{g}(t). \quad (2.4.4)$$

By Proposition 2.4.1, the system (2.4.4) has m linearly independent solutions $\underline{g}_1(\cdot), \dots, \underline{g}_m(\cdot)$ such that as $t \rightarrow \infty$

$$\begin{aligned} \underline{g}_1(t) &\rightarrow \underline{e} \\ e^{\beta t} \underline{g}_k(t) &\rightarrow \underline{0}; \quad k=2, \dots, m \end{aligned}$$

where $0 > -\beta > \operatorname{Re} \lambda_j$ for every eigenvalue λ_j of $T+T^{\circ}A^{\circ}$ for which $\operatorname{Re} \lambda_j < 0$. Now every column of $G(z, t)$ satisfies (2.4.4) whence the i -th column of $G(z, t)$ converges as $t \rightarrow \infty$ to $c_i(z)\underline{e}$ where $c_i(z)$ is a constant which depends on z only. Thus as $t \rightarrow \infty$, $G(z, t)$ converges to a matrix $G(z)$ all whose rows are identical.

Now we can write (2.2.3), (2.2.4) as

$$G(z, t) = I + \int_0^t [(T+T^{\circ}A^{\circ})+(z-1)\{1-\tilde{H}(\tau)\}T^{\circ}A^{\circ}]G(z, \tau)d\tau, \quad t \geq 0$$

and therefore

$$G(z) = I + \int_0^{\infty} [(T+T^{\circ}A^{\circ})+(z-1)\{1-\tilde{H}(\tau)\}T^{\circ}A^{\circ}]G(z, \tau)d\tau.$$

Now,

$$\begin{aligned} G(z) &= \theta G(z) \\ &= \theta + (z-1)\theta T^{\circ}A^{\circ} \int_0^{\infty} \{1-\tilde{H}(\tau)\}G(z, \tau)d\tau \end{aligned} \quad (2.4.5)$$

Noting that $G(1, \tau) = \exp[(T + T^0 A^0) \tau]$ is stochastic for every $\tau \geq 0$, we can apply the Monotone Convergence Theorem to show that as $z \uparrow 1$,

$$\int_0^{\infty} \{1 - \tilde{H}(\tau)\} G(z, \tau) d\tau \underline{e} \uparrow \mu \underline{e}.$$

Thus from (2.4.5) it follows that as $z \uparrow 1$, $G(z) \rightarrow \theta$, i.e., $G(1-) = \theta$, and the proof is complete.

Remarks:

1. Note that the (i, j) -th entry of $G(1-)$ is the stationary probability that the phase of the arrival process is j . Clearly this must be θ_j , for, $\underline{\theta}$ is the invariant probability vector of the Markov Chain Q^* governing the phases.
2. The result above also shows that the rows of $G(z)$ define a proper joint probability distribution.

The following Theorem is essentially a re-statement of Theorem 2.4.3 in the terminology of probability theory.

Theorem 2.4.6: Under the assumption $\mu = \int_0^{\infty} \{1 - \tilde{H}(\tau)\} d\tau < \infty$, the process $\{(X(t), J(t)) : t \geq 0\}$ is ergodic and has a stationary distribution independent of the initial conditions.

Proof: A trivial probabilistic argument shows that even if we assume that $X(0) = k$, $J(0) = i$ for any $k \geq 0$, $1 \leq i \leq m$, then we would still get the same limit $c_j(z)$ obtained in Theorem 2.4.3 for the sum

$$\sum_{n=0}^{\infty} z^n P[X(t)=n, J(t)=j | X(0)=k, J(0)=i]$$

whence the result.

CHAPTER III

MOMENTS OF THE QUEUE LENGTH

3.1 Introduction

In this Chapter we shall be concerned with obtaining the moments of the queue length $X(t)$. Letting $N(t)$ denote the number of arrivals in $(0, t]$, we can easily see that $0 \leq X(t) \leq N(t)$ a.s., and since all the moments of $N(t)$ exist [6], so do the moments of $X(t)$. Let $\mu_i^{(k)}(t)$ denote the k -th factorial moment of $X(t)$ under the assumption $X(0)=0$, $J(0)=i$. That is,

$$\mu_i^{(k)}(t) = E[X^{(k)}(t) | X(0)=0, J(0)=i], \quad k \geq 1, \quad 1 \leq i \leq m,$$

where $X^{(k)}(t)$ denotes the factorial product $X(t)[X(t)-1]\dots[X(t)-k+1]$. We also let $\underline{\mu}^{(k)}(t)$ denote the m -vector whose i -th entry is $\mu_i^{(k)}(t)$, $k \geq 1$.

Applying the rules of Calculus "rather formally" to the basic system of differential equations (2.2.3) and evaluating

$$\frac{\partial^k}{\partial z^k} G(z, t) \underline{e} \Big|_{z=1-}, \quad \text{we easily obtain}$$

$$\left. \begin{aligned} \frac{d}{dt} \underline{\mu}^{(1)}(t) &= (T + T \circ A^\circ) \underline{\mu}^{(1)}(t) + \{1 - \tilde{H}(t)\} T \circ A^\circ \underline{e} \\ \underline{\mu}^{(1)}(0) &= \underline{0} \end{aligned} \right\} \quad (3.1.1)$$

and for $k \geq 2$

$$\left. \begin{aligned} \frac{d}{dt} \underline{\mu}^{(k)}(t) &= (T + T \circ A \circ) \underline{\mu}^{(k)}(t) + k \{1 - \tilde{H}(t)\} T \circ A \circ \underline{\mu}^{(k-1)}(t) \\ \underline{\mu}^{(k)}(0) &= \underline{0}. \end{aligned} \right\} (3.1.2)$$

The above systems of differential equations can easily be solved numerically to obtain the first few moments of the queue size. In Section 3.4 we shall present some sample computations of the first four moments with as many as ten phases.

The rigorous proof of (3.1.1) and (3.1.2) which involves the application of a number of well-known theorems in real analysis is presented in the next section. Section 3.3 contains a number of remarks on the numerical computation of the probability distribution and the moments of the queue length. Finally in Section 3.4 we present some sample computations of the first four moments of the queue length for models with as many as ten phases.

3.2 Moments of the Queue Length

We now state a few well-known results from real analysis which we will need in the course of proving (3.1.1) and (3.1.2).

Lemma 3.2.1: Let $(\Omega, \mathcal{B}, \phi)$ be a measure space, and let $\{X_z: z \in [a, b]\}$, where $[a, b]$ is a finite closed interval of \mathbb{R}^1 , be integrable functions from Ω to \mathbb{R}^1 . If $\frac{dX_z}{dz}$ exists on $[a, b]$ and $|\frac{dX_z}{dz}| \leq Y$ where Y is integrable, then

$$\frac{d}{dz} \int_{\Omega} \chi_z d\phi = \int_{\Omega} \frac{d\chi_z}{dz} d\phi \quad \text{for all } z \in [a, b].$$

Proof: See p. 126, 3⁰, Loève [4].

Proposition 3.2.2: Let $(\Omega, \mathcal{B}, \phi)$ be a measure space and let $\{\chi_z: z \in [a, b]\}$ be integrable functions from Ω to \mathbb{R}^m . If for all $k \geq 1$, $\frac{d^k \chi_z}{dz^k}$ exists on $[a, b]$ and $|\frac{d^k \chi_z}{dz^k}| \leq \gamma_k$ where γ_k is integrable, then

$$\frac{d^k}{dz^k} \int_{\Omega} \chi_z d\phi = \int_{\Omega} \frac{d^k \chi_z}{dz^k} d\phi, \quad k \geq 1.$$

Proof: This Proposition follows readily by repeated application of Lemma 3.2.1 and mathematical induction.

Proposition 3.2.3: If f is Lebesgue integrable on $[a, b]$ and

$$F(x) = \int_a^x f(\tau) d\tau, \quad x \in [a, b],$$

then

a) $F(\cdot)$ is a continuous function of bounded variation on $[a, b]$.

b) $F'(x) = f(x)$ a.e. on $[a, b]$.

Proof: For Part (a) we refer the reader to Lemma 6, p. 87, Royden [9]. Part (b) is Theorem 9, p. 89, Royden [9].

We are now ready to prove

Theorem 3.2.4: For all $t \in [0, \infty)$,

$$\underline{\mu}^{(1)}(t) = (T + T \circ A^0) \int_0^t \underline{\mu}^{(1)}(\tau) d\tau + \int_0^t \{1 - \tilde{H}(\tau)\} d\tau \underline{1}^0 \quad (3.2.5)$$

and for all $k \geq 2$,

$$\underline{\mu}^{(k)}(t) = (T + T^\circ A^\circ) \int_0^t \underline{\mu}^{(k)}(\tau) d\tau + k \int_0^t \{1 - \tilde{H}(\tau)\} T^\circ A^\circ \underline{\mu}^{(k-1)}(\tau) d\tau \quad (3.2.6)$$

Proof: From (2.2.3) and (2.2.4) it follows easily that

$$G(z, t) = I + \int_0^t [(T + T^\circ A^\circ) + (z-1)\{1 - \tilde{H}(\tau)\} T^\circ A^\circ] G(z, \tau) d\tau,$$

whence

$$\underline{\mu}^{(k)}(t) = \left[\frac{\partial^k}{\partial z^k} \int_0^t [(T + T^\circ A^\circ) + (z-1)\{1 - \tilde{H}(\tau)\} T^\circ A^\circ] G(z, \tau) d\tau \underline{e} \right]_{z=1-} \quad (3.2.7)$$

For $z \in [0, 1]$, $\tau \in [0, t]$, clearly,

$$\begin{aligned} 0 &\leq \left[\frac{\partial^k}{\partial z^k} G(z, \tau) \underline{e} \right]_i \\ &\leq \left[\frac{\partial^k}{\partial z^k} G(z, \tau) \underline{e} \right]_{z=1-} = \mu_i^{(k)}(\tau) \end{aligned}$$

$$\leq E[N^{(k)}(\tau) | J(0) = i]$$

$$\leq E[N^{(k)}(t) | J(0) = i],$$

where $N^{(k)}(t) = N(t)[N(t)-1]\dots[N(t)-k+1]$. The last inequality in the above chain of inequalities is got by using the fact that $N_k(\cdot) \nearrow$ a.s. Now, denoting $E[N^{(k)}(\tau) | J(0) = i]$ by $v_i^{(k)}(\tau)$, we have

$$0 \leq \int_0^t v_i^{(k)}(t) d\tau < \infty \quad \text{for every } 1 \leq i \leq m, k \geq 1.$$

Also for $0 \leq z \leq 1$, $0 \leq \tau \leq t$,

$$\begin{aligned}
& \left| \left| \frac{\partial^k}{\partial z^k} [((T+T^\circ A^\circ) + (z-1)\{1-\tilde{H}(\tau)\})T^\circ A^\circ] G(z, \tau) \underline{e} \right| \right| \\
&= \left| \left| [(T+T^\circ A^\circ) + (z-1)\{1-\tilde{H}(\tau)\})T^\circ A^\circ] \frac{\partial^k}{\partial z^k} G(z, \tau) \underline{e} \right. \right. \\
&\quad \left. \left. + k\{1-\tilde{H}(\tau)\})T^\circ A^\circ \frac{\partial^{k-1}}{\partial z^{k-1}} G(z, \tau) \underline{e} \right| \right|
\end{aligned}$$

$$\leq \{ ||T+T^\circ A^\circ|| + ||T^\circ A^\circ|| \} ||\underline{v}^{(k)}(t)|| + k ||T^\circ A^\circ|| ||\underline{v}^{(k-1)}(t)||,$$

where, $\underline{v}^{(k)}(t)$ is the vector whose i -th component is $v_i^{(k)}(t)$.

Now, in Proposition 3.2.2 set $\Omega=[0, t]$, B =Borel subsets of $[0, t]$, ϕ =Lebesgue measure,

$$\underline{X}_z(\tau) = [(T+T^\circ A^\circ) + (z-1)\{1-\tilde{H}(\tau)\})T^\circ A^\circ] G(z, \tau) \underline{e}$$

$$\underline{Y}_k(\tau) = (||T+T^\circ A^\circ|| + ||T^\circ A^\circ||) ||\underline{v}^{(k)}(t)|| + k ||T^\circ A^\circ|| ||\underline{v}^{(k-1)}(t)||$$

to obtain, using (3.2.7), that

$$\underline{u}^{(k)}(t) = \int_0^t \frac{\partial^k}{\partial z^k} [((T+T^\circ A^\circ) + (z-1)\{1-\tilde{H}(\tau)\})T^\circ A^\circ] G(z, \tau) \underline{e} \Big|_{z=1-} d\tau,$$

and the theorem follows.

Corollary 3.2.8:

a) For all $k \geq 1$, $\underline{u}^{(k)}(\cdot)$ is continuous and of bounded variation in any finite interval $[0, t]$.

b) We also have

$$\frac{d}{dt} \underline{u}^{(1)}(t) = (T+T^\circ A^\circ) \underline{u}^{(1)}(t) + \{1-\tilde{H}(t)\} T^\circ A^\circ \underline{e}, \quad \text{a.e.}$$

$$\underline{u}^{(1)}(0) = \underline{0}$$

and for $k \geq 2$

$$\frac{d}{dt} \underline{u}^{(k)}(t) = (T+T^\circ A^\circ) \underline{u}^{(k)}(t) + k\{1-\tilde{H}(t)\} T^\circ A^\circ \underline{u}^{(k-1)}(t), \quad \text{a.e.}$$

$$\underline{u}^{(k)}(0) = \underline{0}$$

Proof: This Corollary is an immediate consequence of Theorem 3.2.4 and Proposition 3.2.3.

The above discussion completes the proof of (3.1.1) and (3.1.2) which were derived heuristically at the beginning of this Chapter. In a very easy way we can now prove

Theorem 3.2.9: Under the assumption $\mu = \int_0^\infty \{1 - \tilde{H}(\tau)\} d\tau < \infty$, for any $k \geq 1$ the vector of factorial moments $\underline{\mu}^{(k)}(t)$ converges as $t \rightarrow \infty$ to a (finite) vector all whose components are equal. Further for any $k \geq 1$, $\underline{\mu}^{(k)}(\cdot)$ is bounded in $[0, \infty)$.

Proof: Clearly it suffices to prove the existence and finiteness of these limits. To this end note that (3.2.5) implies

$$\underline{\mu}^{(1)}(t) = \underline{\mu} \int_0^t \{1 - \tilde{H}(\tau)\} d\tau \rightarrow \underline{\mu} \int_0^\infty \{1 - \tilde{H}(\tau)\} d\tau < \infty \text{ as } t \rightarrow \infty.$$

This shows that $\underline{\mu}^{(1)}(t)$ converges to a finite limit as $t \rightarrow \infty$ and also (since $\underline{\mu}_1(\cdot)$ is continuous in $[0, \infty)$) that $\underline{\mu}_1(\cdot)$ is bounded in $[0, \infty)$. Now as induction hypothesis assume that the Theorem is true for $k-1$. From (3.2.6) we now have

$$\underline{\mu}^{(k)}(t) = \underline{\mu} \int_0^t \{1 - \tilde{H}(\tau)\} \underline{\mu}^{(k-1)}(\tau) d\tau,$$

and the above integral converges as $t \rightarrow \infty$ to a finite limit since $\underline{\mu}^{(k-1)}(\cdot)$ is bounded on $[0, \infty)$. Now the boundedness of $\underline{\mu}^{(k)}(\cdot)$ follows from its continuity. By mathematical induction, the proof is complete.

An interesting special case of the PH/G/ ∞ queue is the one in which the PH-Renewal process describing arrivals is stationary - i.e. where the initial phase is chosen according to the vector $\underline{\theta}$. For this case we have the interesting

Theorem 3.2.10: Suppose the initial phase $J(0)$ is chosen according to the vector $\underline{\theta}$, i.e. the PH-Renewal process is stationary. Then

- a) the mean system size at t is given by $\lambda \int_0^t \{1 - \tilde{H}(\tau)\} d\tau$.
- b) if $\mu = \int_0^\infty \{1 - \tilde{H}(\tau)\} d\tau < \infty$, then the mean of the steady state distribution of the system size is $\lambda\mu$ where λ^{-1} is the mean inter-arrival time.

Proof: Pre-multiplying (3.2.5) by $\underline{\theta}$,

$$\underline{\theta} \underline{\mu}^{(1)}(t) = \underline{\theta} \underline{I}^0 \int_0^t \{1 - \tilde{H}(\tau)\} d\tau,$$

and the result in Part (a) follows by noting that $\underline{\theta} = -\lambda \underline{\alpha} \underline{T}^{-1}$ and $\underline{I}^0 = -\underline{T} \underline{\theta}$. Part (b) now easily follows from Part (a).

Using the above theorem and Theorem 3.2.9 we immediately obtain

Corollary 3.2.11: If $\mu = \int_0^\infty \{1 - \tilde{H}(\tau)\} d\tau < \infty$, then

$$\underline{\mu}^{(1)}(t) \rightarrow \lambda \mu \underline{e} \quad \text{as } t \rightarrow \infty.$$

3.3 Some Remarks on Computational Methods

Except in the special case of Poisson arrivals, there does not seem to be any hope of obtaining in closed form the distribution of the system size or the moments thereof for

the PH/G/ ∞ queue. Nevertheless, all the systems of differential equations above lend themselves readily to computations using numerical methods.

For the purpose of computing the moments up to an index k , it appears best to consider the k m differential equations given by (3.1.1) and (3.1.2) as forming a single system. Using a general purpose inter-active software system called DELSIM, due to Professor D. E. Lamb and available at the University of Delaware Computing Center, we computed the time-dependent solutions to the moment-vectors up to the fourth moment for models involving as many as ten phases by applying the Fifth-order Kutta-Merson method. Some of these computations are presented in graphical form in the next section. While the process times for these examples ranged from one half to seven minutes, we point out that the DELSIM system, due to its general nature, does not take into account the nice structure of the system of differential equations at hand. A program which takes into account the special structure of the differential equations (3.1.1), (3.1.2) could handle much larger examples and would also result in considerable savings in computer time and storage.

Defining

$$\underline{g}_k(t) = G_k(t) \underline{e}, \quad k \geq 0$$

we note that the i -th entry of $\underline{g}_k(t)$ is given by

$$g_{ki}(t) = P[X(t)=k | X(0)=0, J(0)=i].$$

It is easily seen from (2.2.3) and (2.2.4) that

$$\left. \begin{aligned} \frac{d}{dt} \underline{g}_0(t) &= [T + \tilde{H}(t)T^{\circ}A^{\circ}] \underline{g}_0(t) \\ \underline{g}_0(0) &= \underline{e} \end{aligned} \right\} \quad (3.3.1)$$

and for $k \geq 1$

$$\left. \begin{aligned} \frac{d}{dt} \underline{g}_k(t) &= [T + \tilde{H}(t)T^{\circ}A^{\circ}] \underline{g}_k(t) + \{1 - \tilde{H}(t)\} T^{\circ}A^{\circ} \underline{g}_{k-1}(t) \\ \underline{g}_k(0) &= \underline{0}. \end{aligned} \right\} \quad (3.3.2)$$

The infinite system of differential equations above needs to be truncated at a sufficiently large value of the index k before any numerical method can be implemented to solve it. It does not appear tractable to develop any optimal methods for such truncation. In the absence of such criteria it appears practical to truncate the system at $k = \lambda\mu + 3\sigma$ where σ is the standard deviation of the stationary distribution. Using such methods we are confident that the equations (3.3.1), (3.3.2) can be solved for queues for which the value of $\lambda\mu$ is even moderately large. But as $\lambda\mu$ becomes very large, say over 200, the systems of differential equations above could become stiff and pose considerable difficulty in solving them numerically. Further work along these lines is under way and will be the subject matter of a forthcoming paper.

3.4 Computational Examples

To illustrate the computations of the moments of the queue length, we considered the following five PH-distributions for the inter-arrival times:

$$\text{Exponential (50),} \quad (3.4.1)$$

$$E(10, 500), \quad (3.4.2)$$

$$0.2E(5, 62.5)+0.8E(5, 1000), \quad (3.4.3)$$

$$0.8E(2, 400)+0.2E(8, 100), \quad (3.4.4)$$

$$0.5E(5, 156.25)+0.5E(5, 625), \quad (3.4.5)$$

where $E(n, \alpha)$ denotes the Erlang distribution with density

$$f(x) = \frac{\alpha^n}{\Gamma(n)} e^{-\alpha x} x^{n-1}, \quad x \geq 0.$$

While each of these PH-distributions has the same mean 0.02, the distributions are qualitatively very different as the graphs of their density functions (Figures 1-7) show. The variances of these distributions are respectively 4×10^{-4} , 0.4×10^{-4} , 11×10^{-4} , 10.7×10^{-4} and 2.528×10^{-4} .

50.0

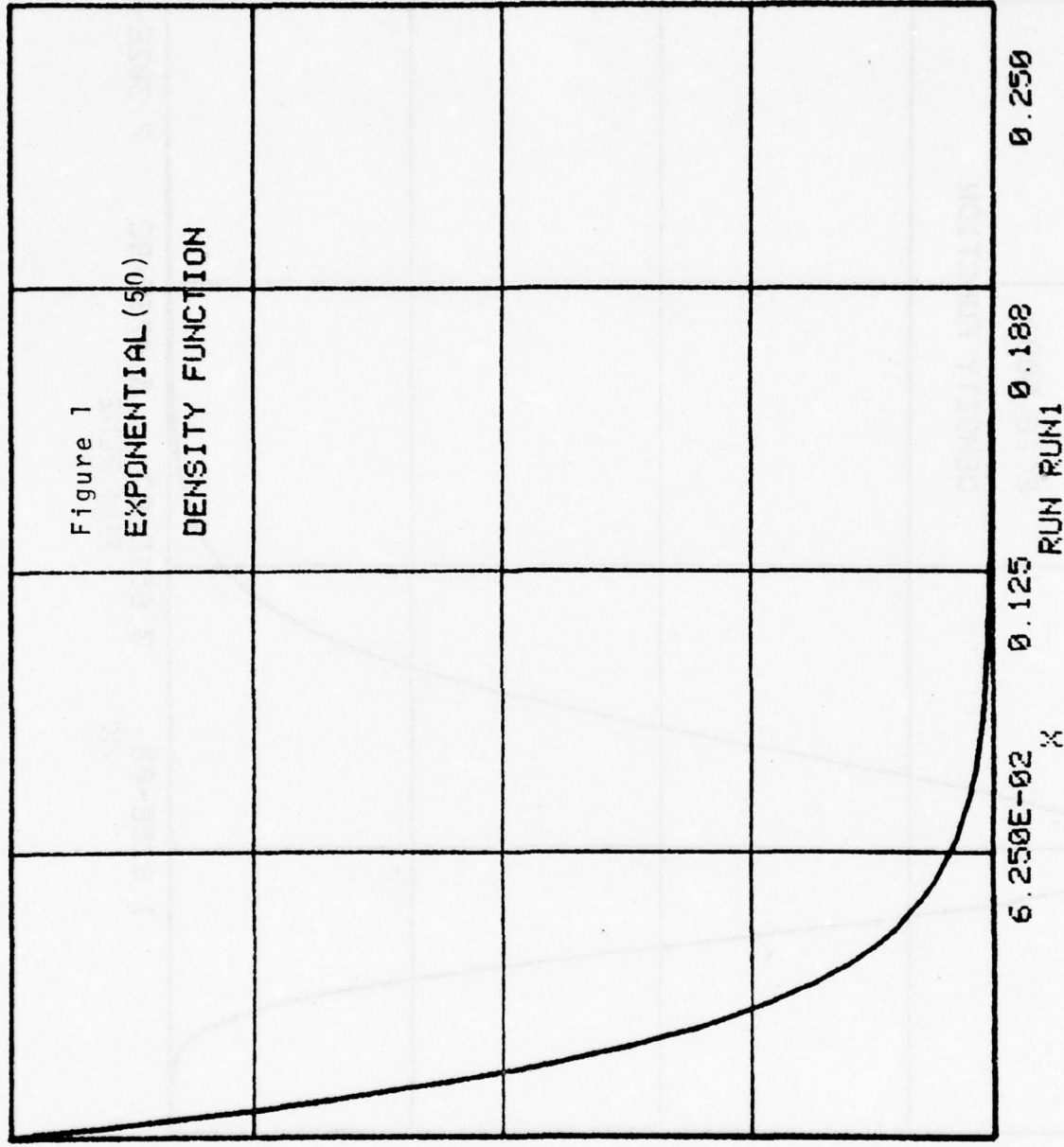
37.5

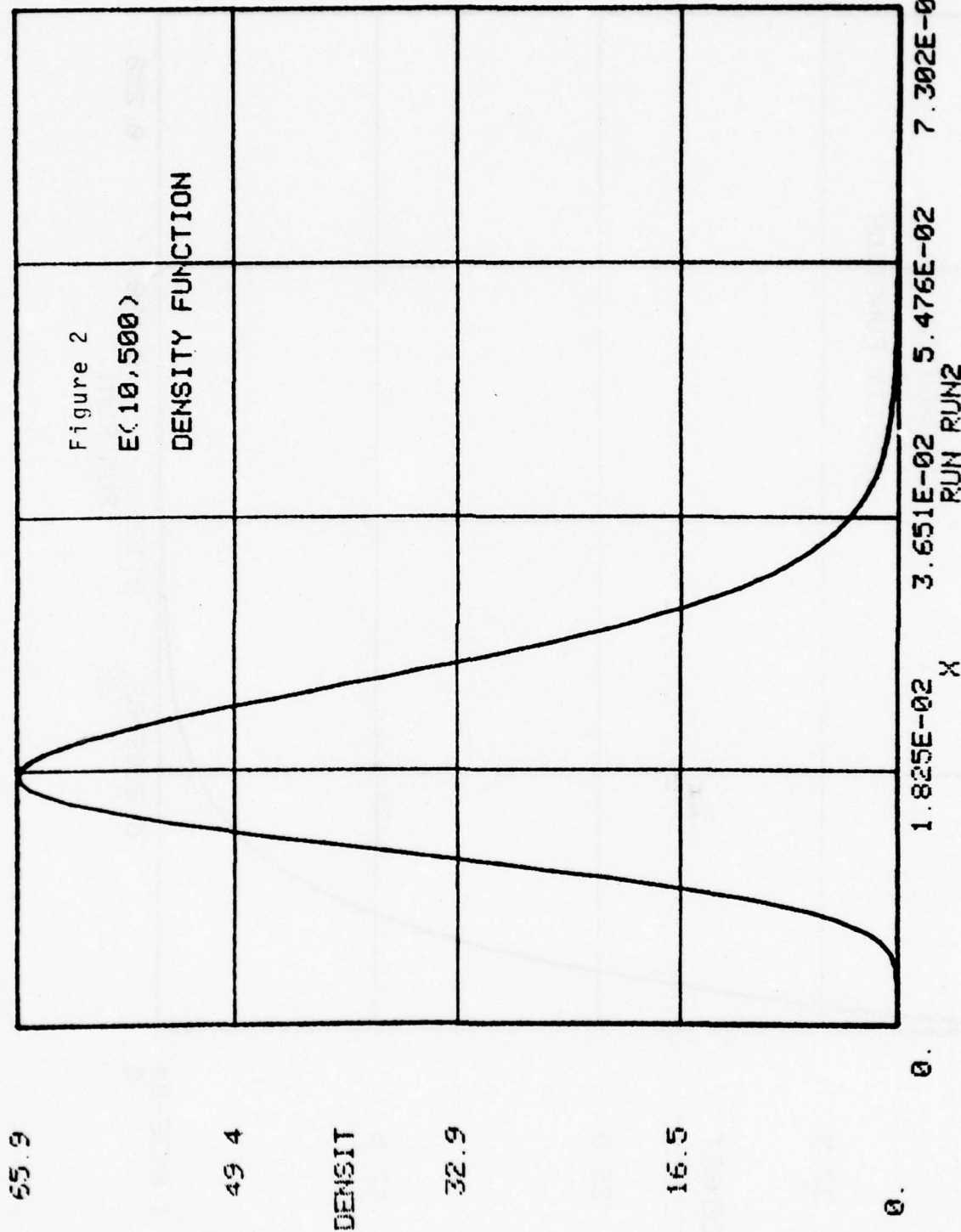
DENSIT

25.0

12.5

1.863E-04
0.





156.

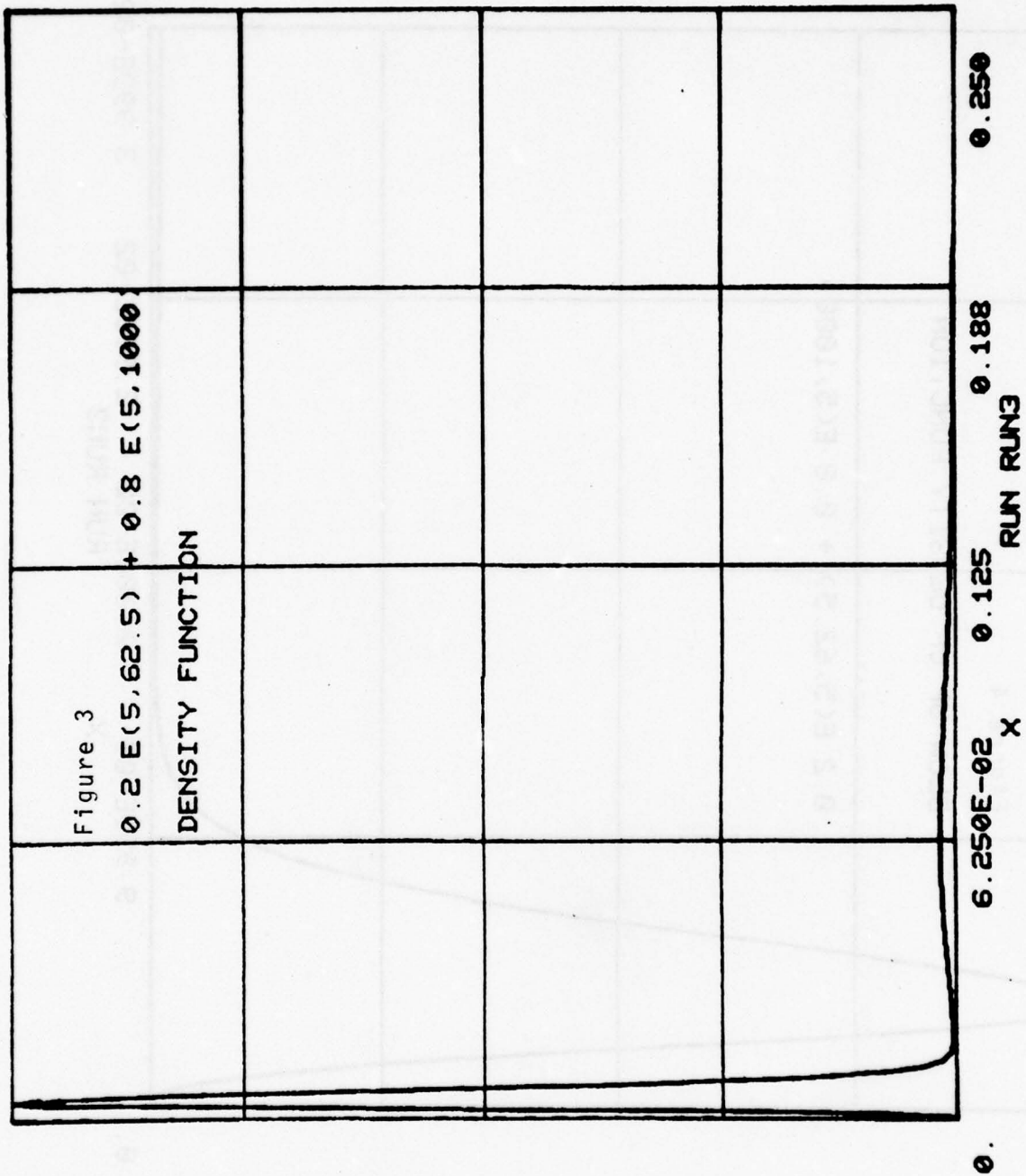
117.

DENSIT

78.1

39.1

0.



156.

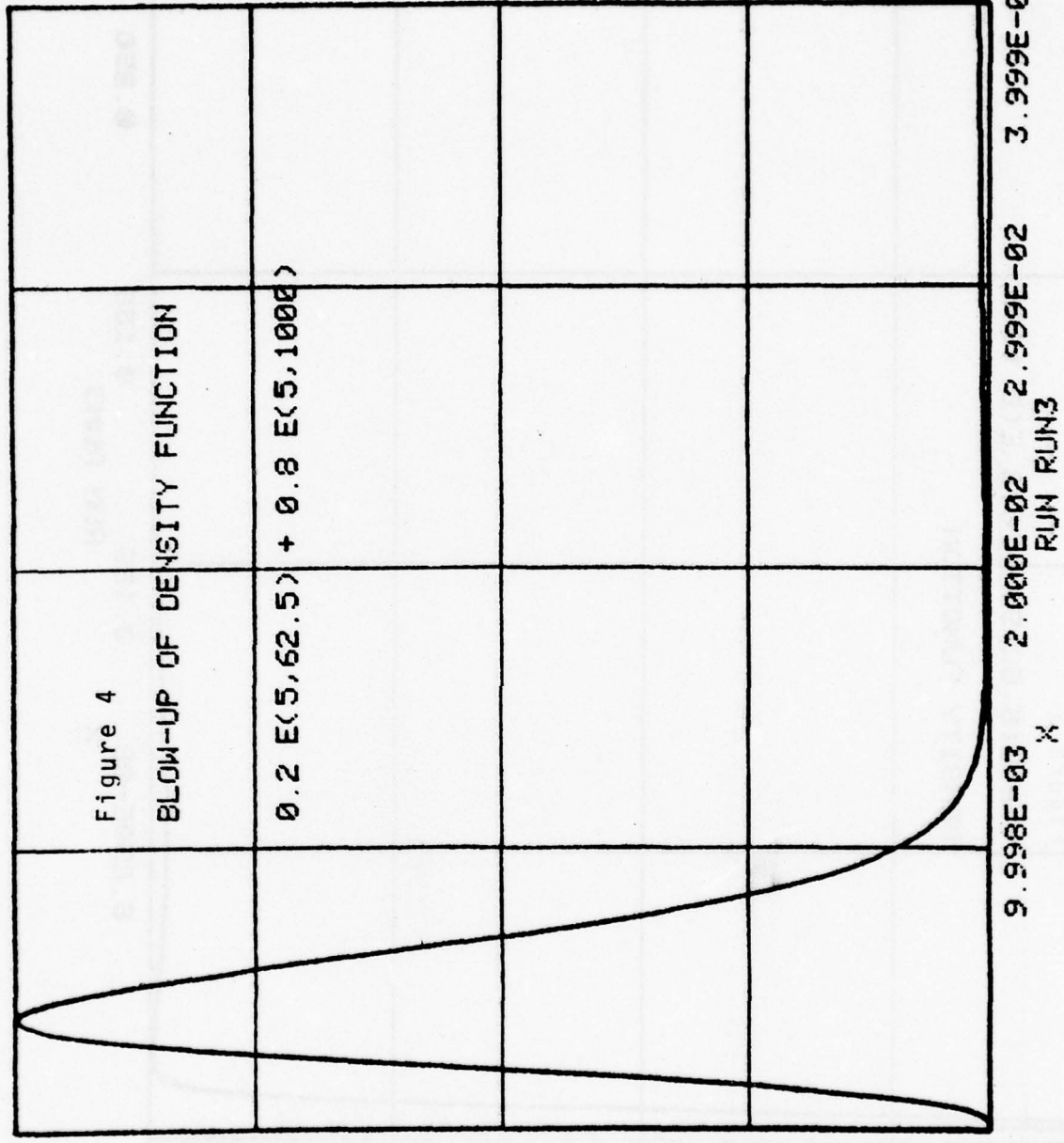
117.

DENSIT

78.1

39.1

0.



4.00

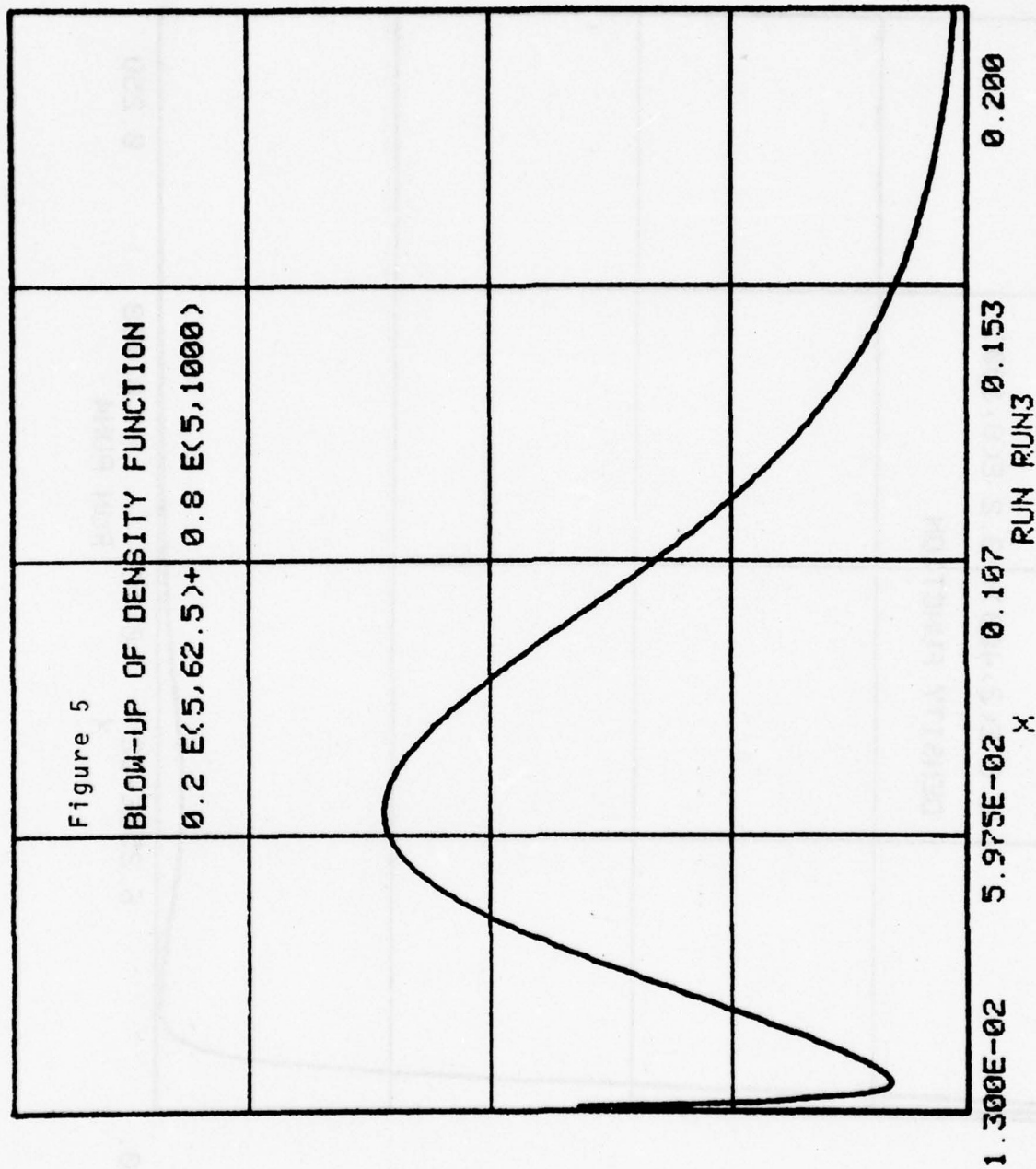
3.00

DENSIT

2.00

1.00

0.



118.

28.3

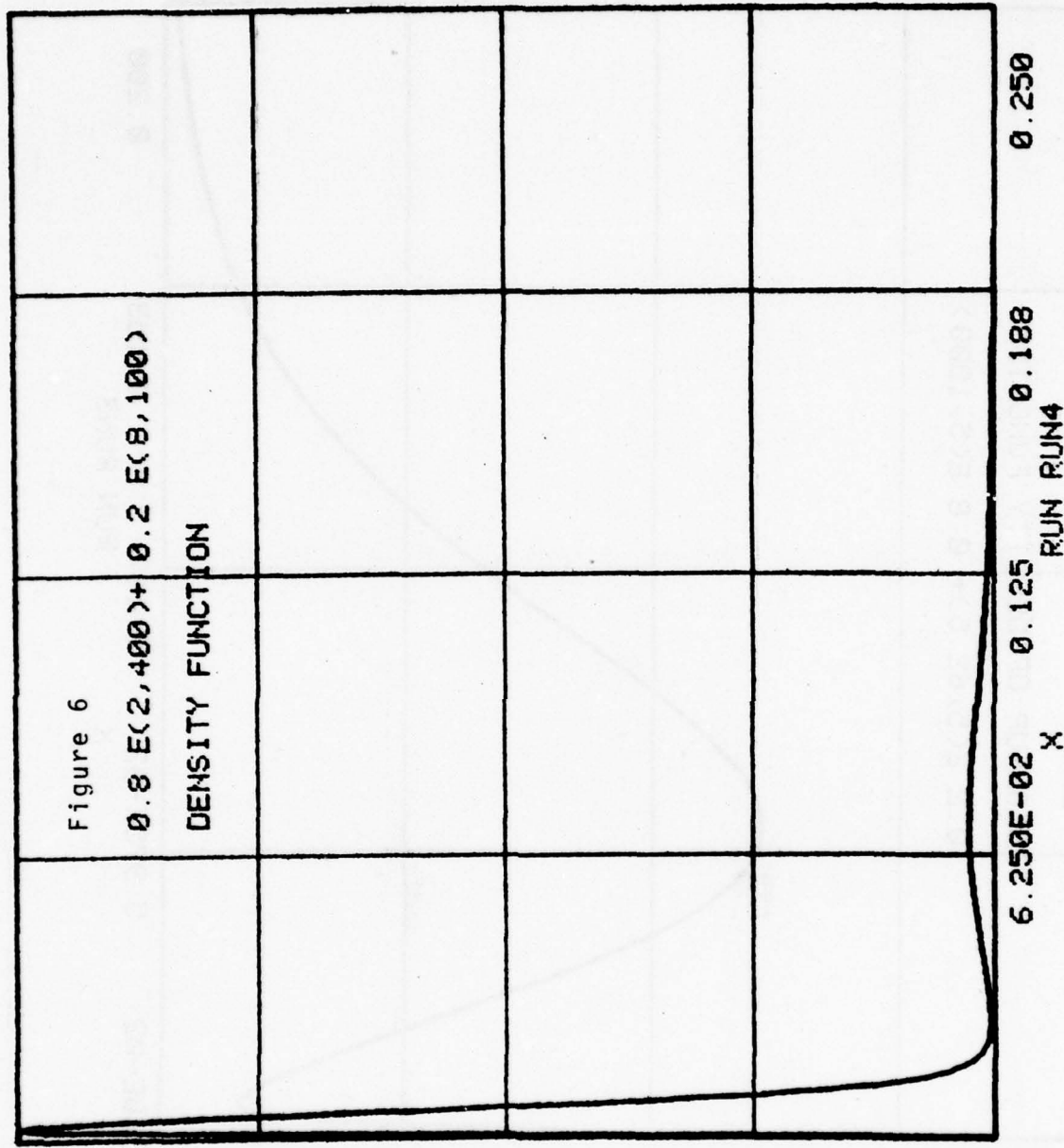
DENSIT

58.9

29.4

0.

0.



0.2 3

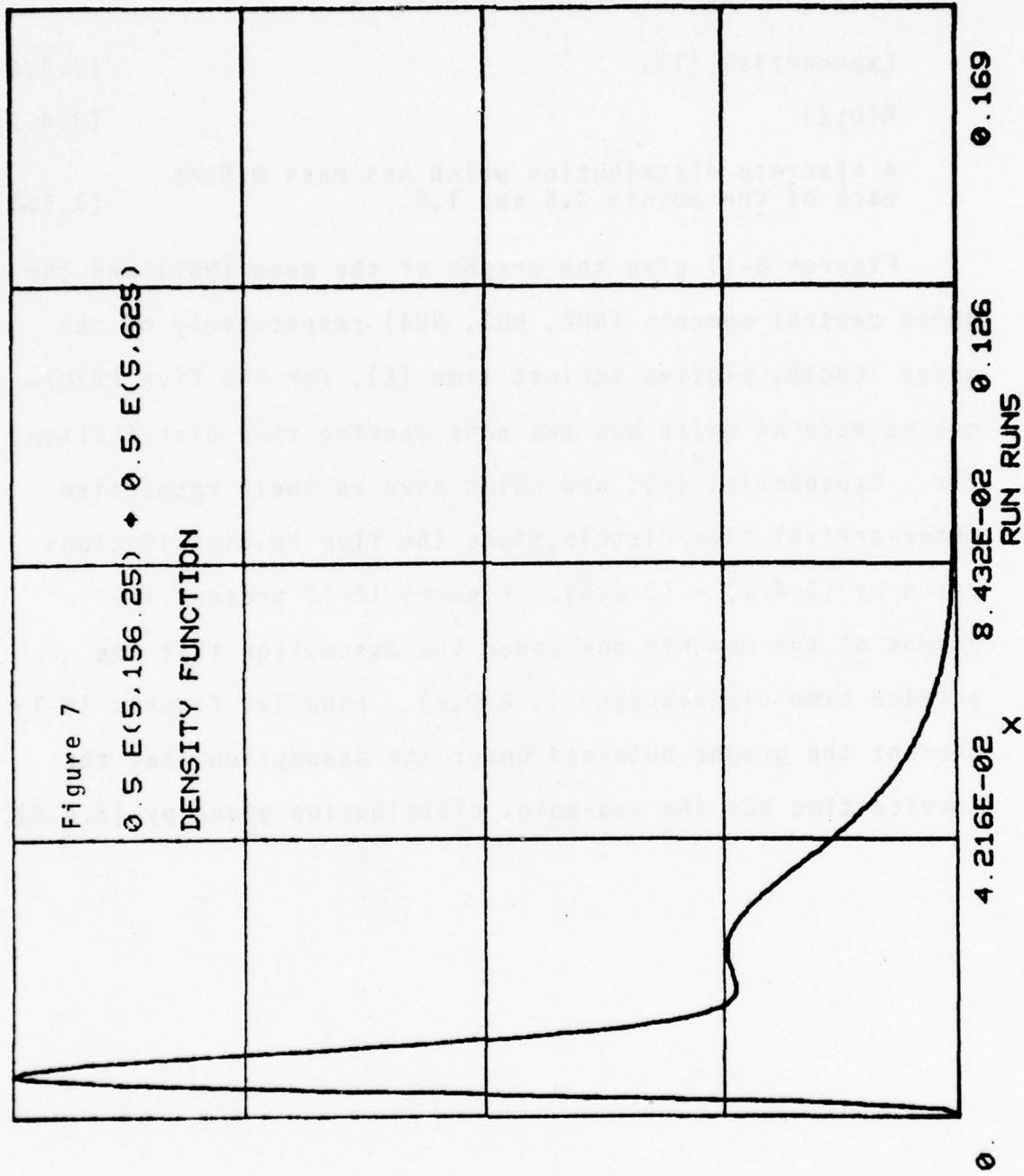
46.7

DENSIT

31.1

15.6

0



For the service time we considered the following three distributions each of which has mean 1:

Exponential (1), (3.4.6)

$R(0,2)$ (3.4.7)

A discrete distribution which has mass 0.5 at each of the points 0.5 and 1.5. (3.4.8)

Figures 8-11 give the graphs of the mean (NU_1) and the three central moments (NU_2 , NU_3 , NU_4) respectively of the queue length, plotted against time (X), for the five PH/G/ ∞ queues each of which has the same service time distribution, viz., Exponential (1), and which have as their respective inter-arrival time distributions the five PH-distributions given by (3.4.1) - (3.4.5). Figures 12-15 present the graphs of the moments now under the assumption that the service time distribution is $R(0,2)$. Finally, Figures 16-19 present the graphs obtained under the assumption that the service time has the two-point distribution given by (3.4.8).

Figure 8

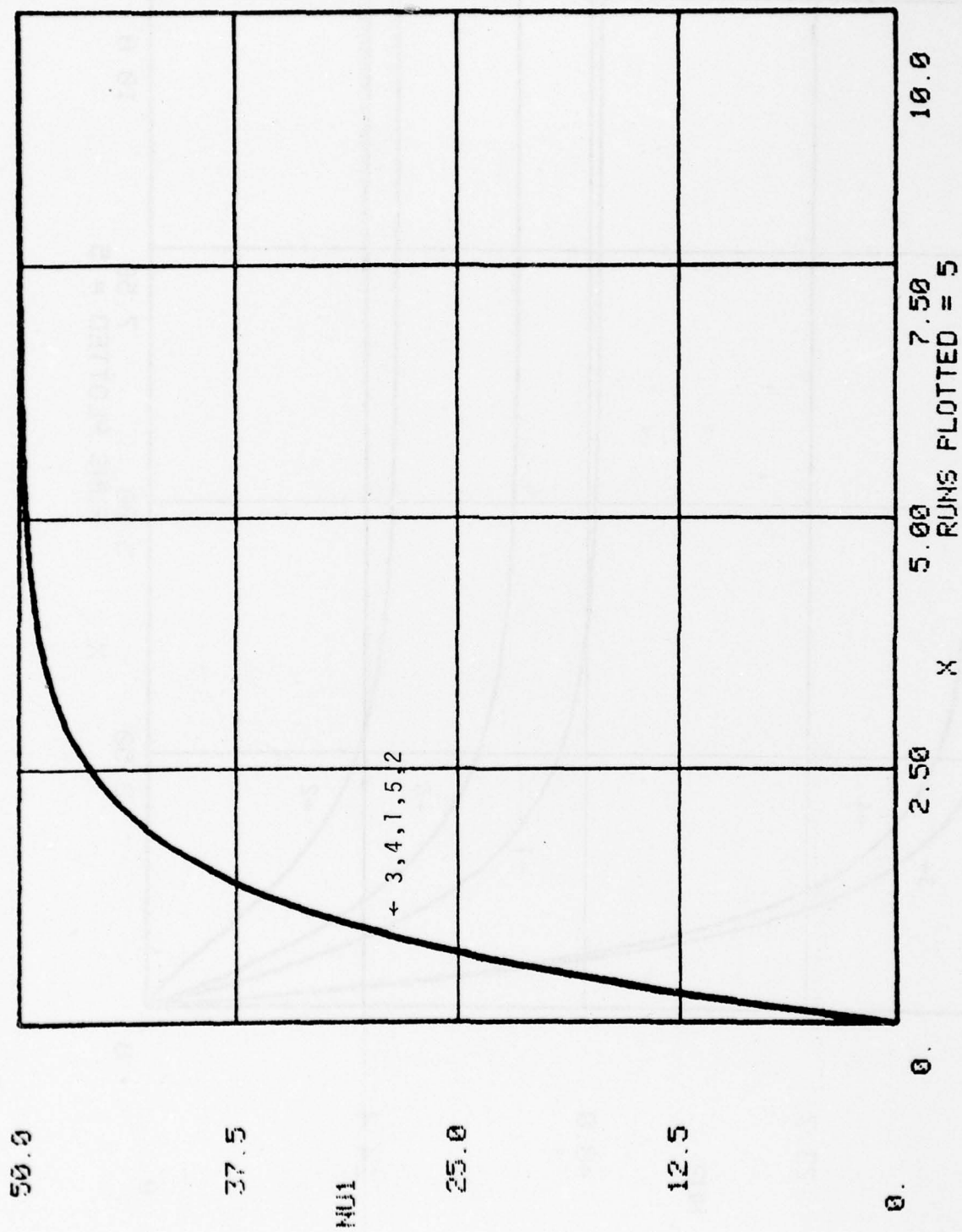


Figure 9

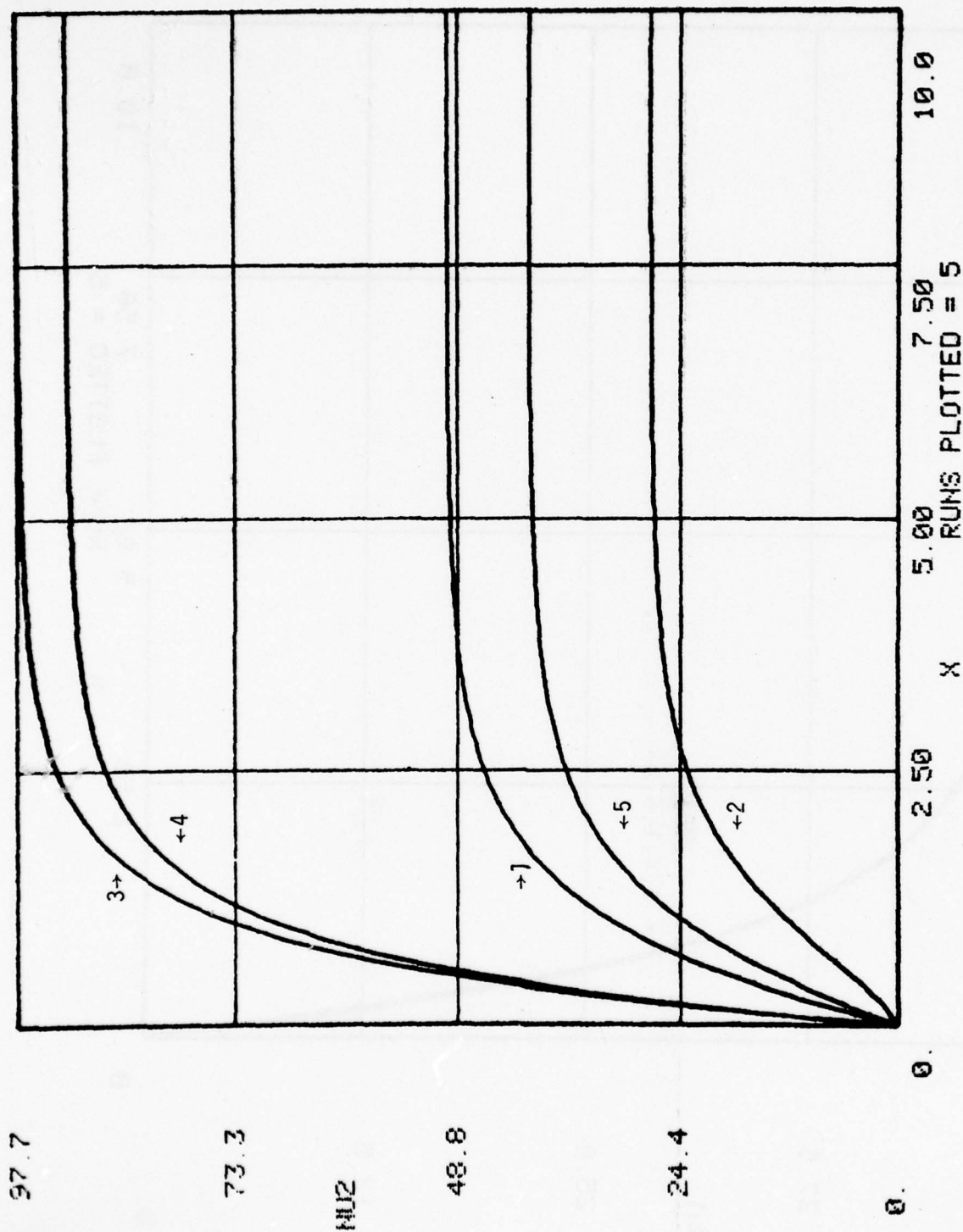


Figure 10

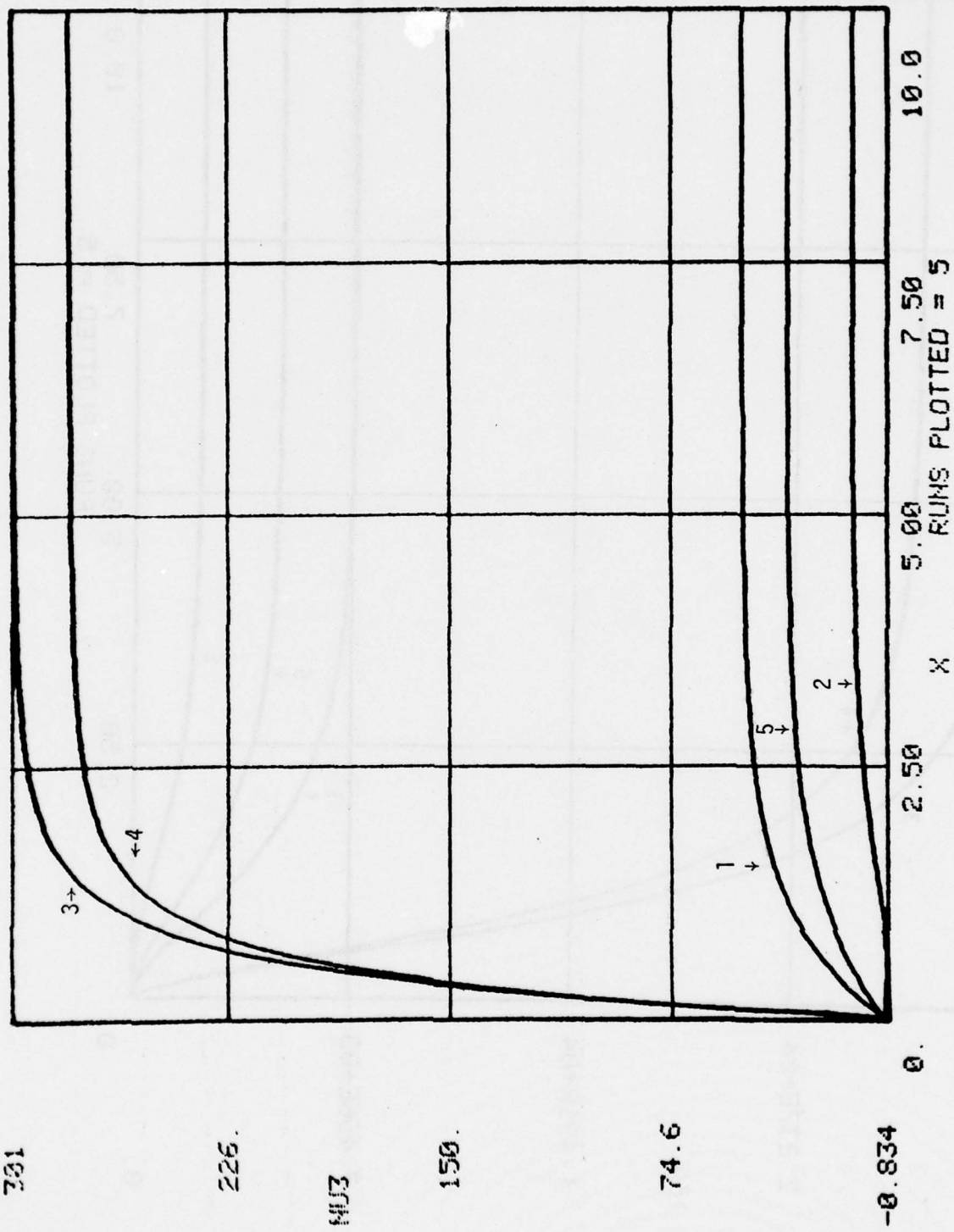


Figure 11

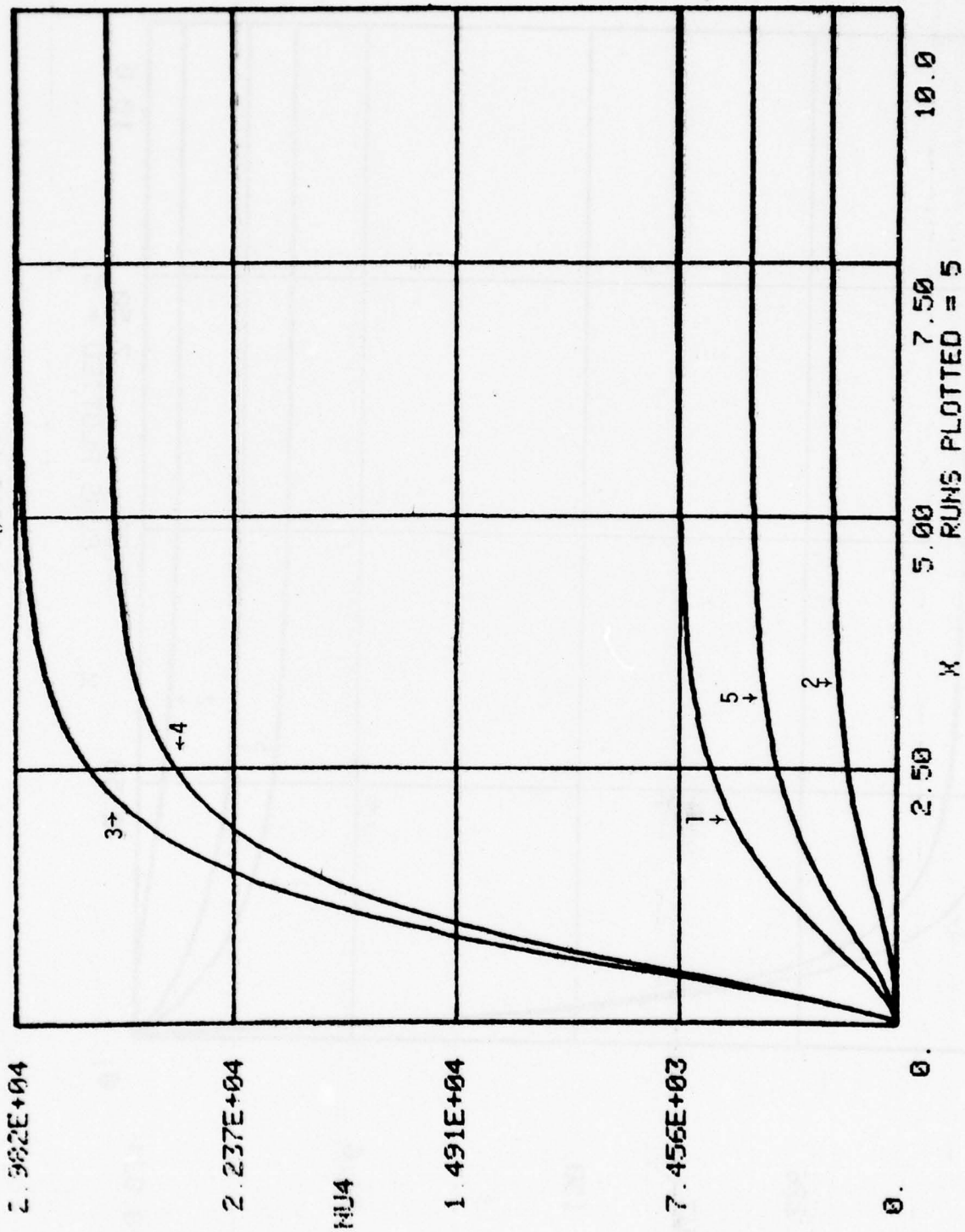


Figure 12

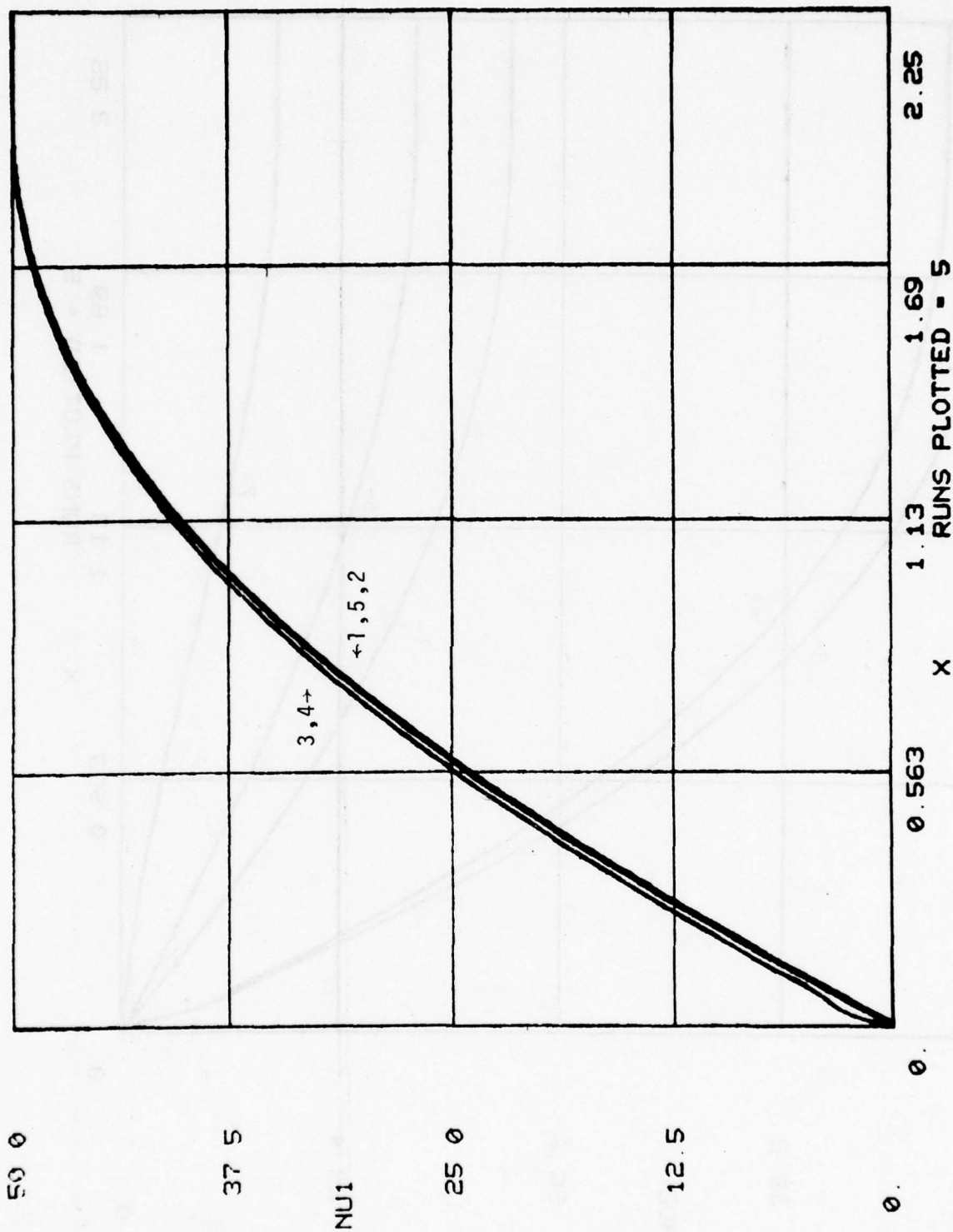


Figure 13

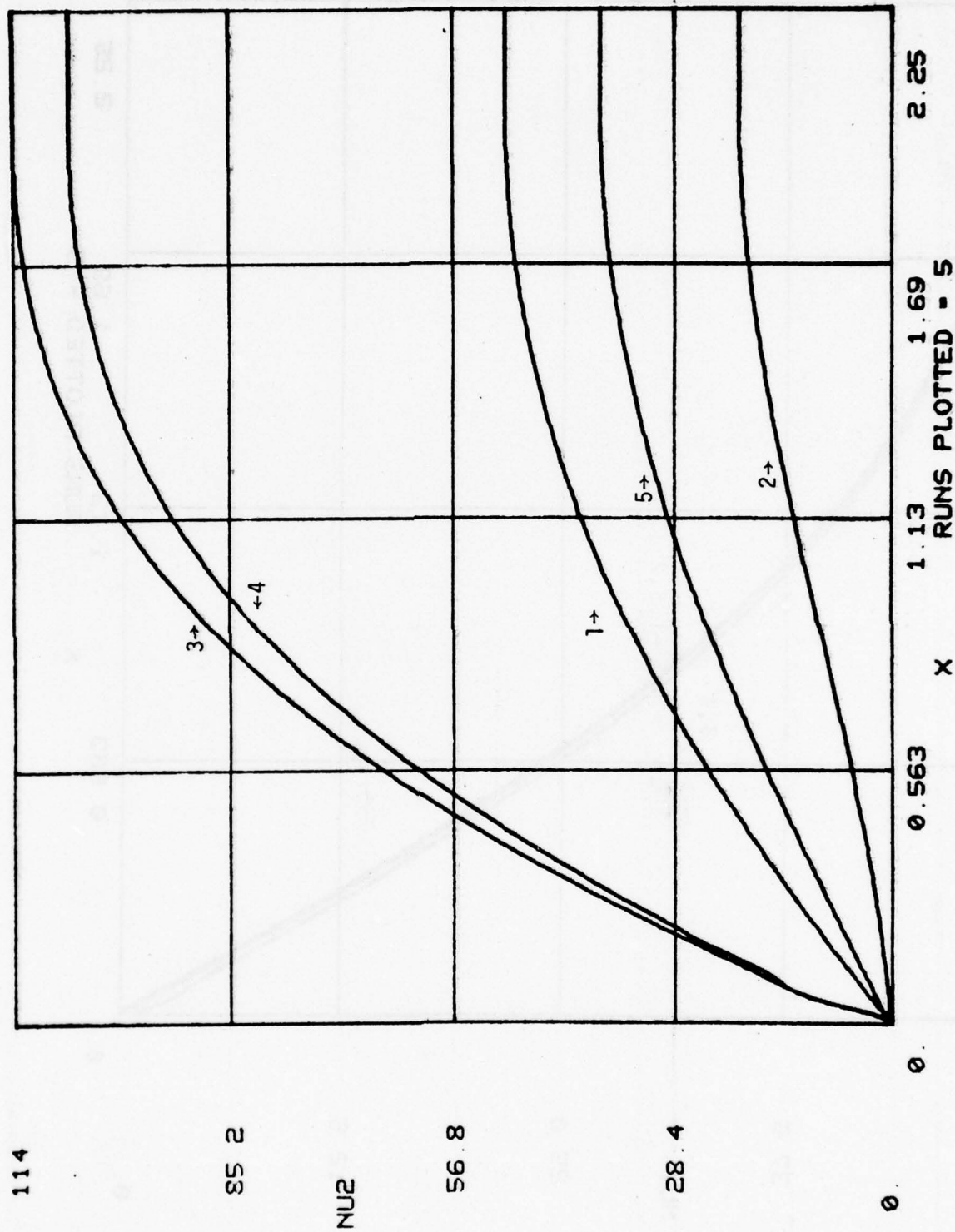


Figure 14

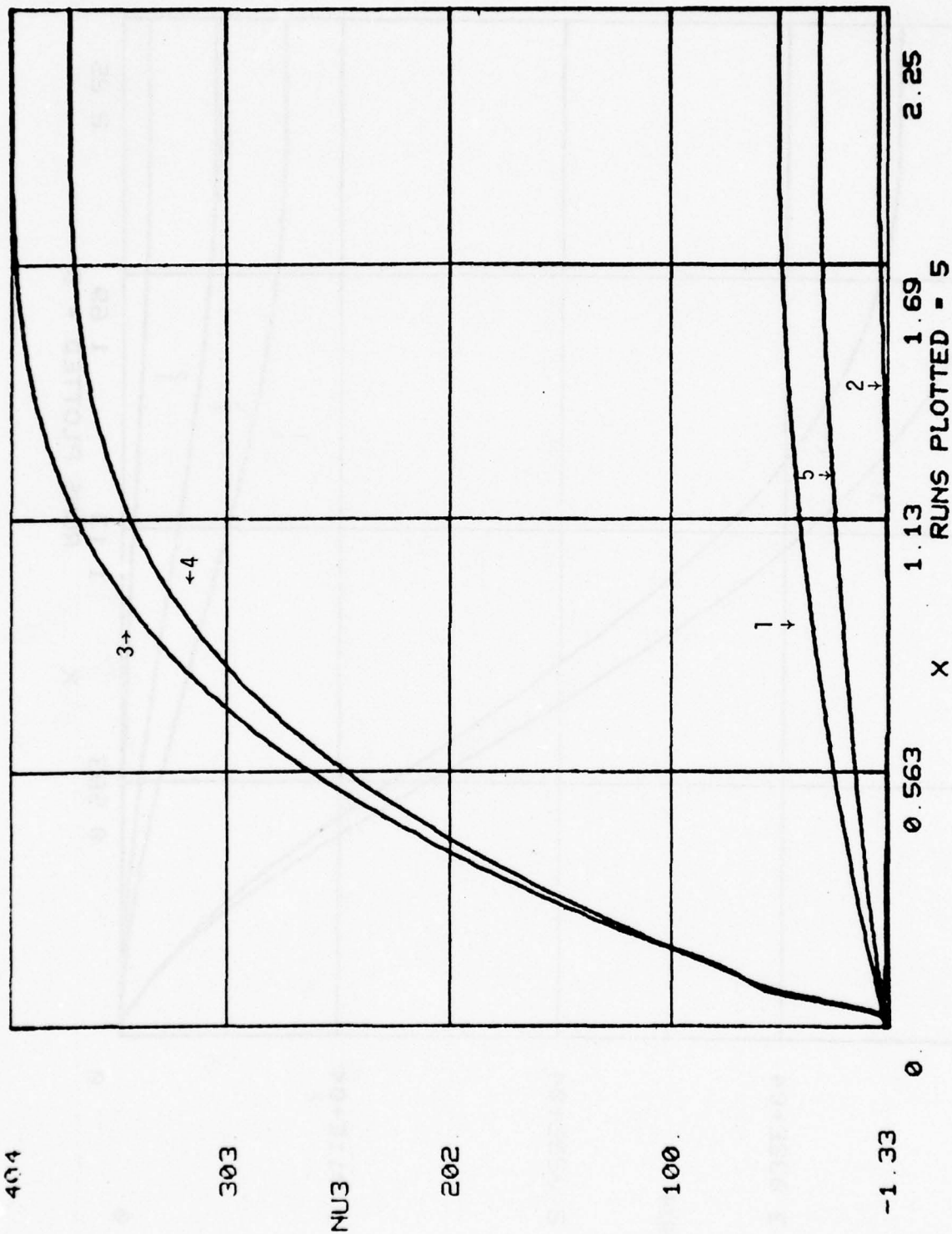


Figure 15

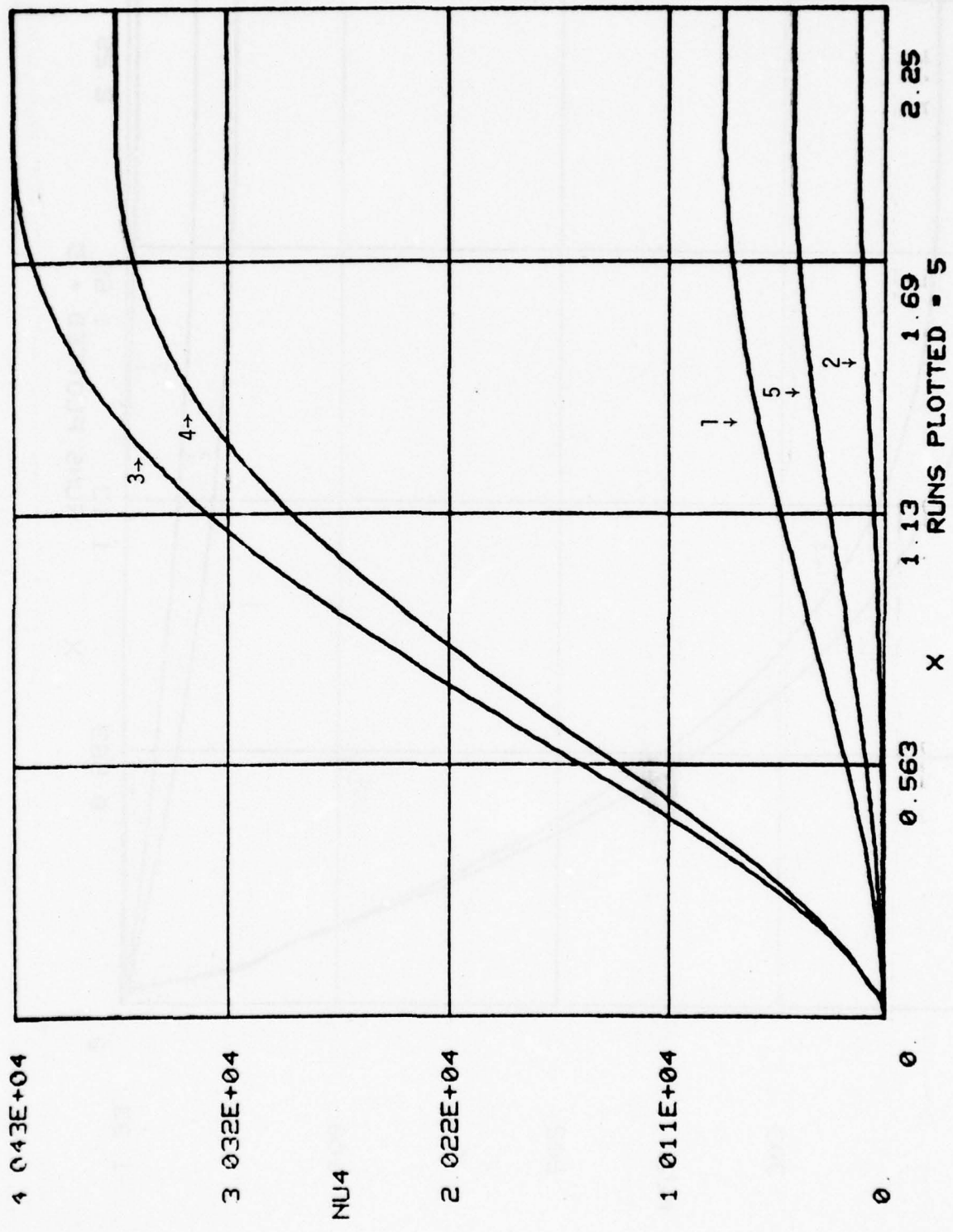


Figure 16

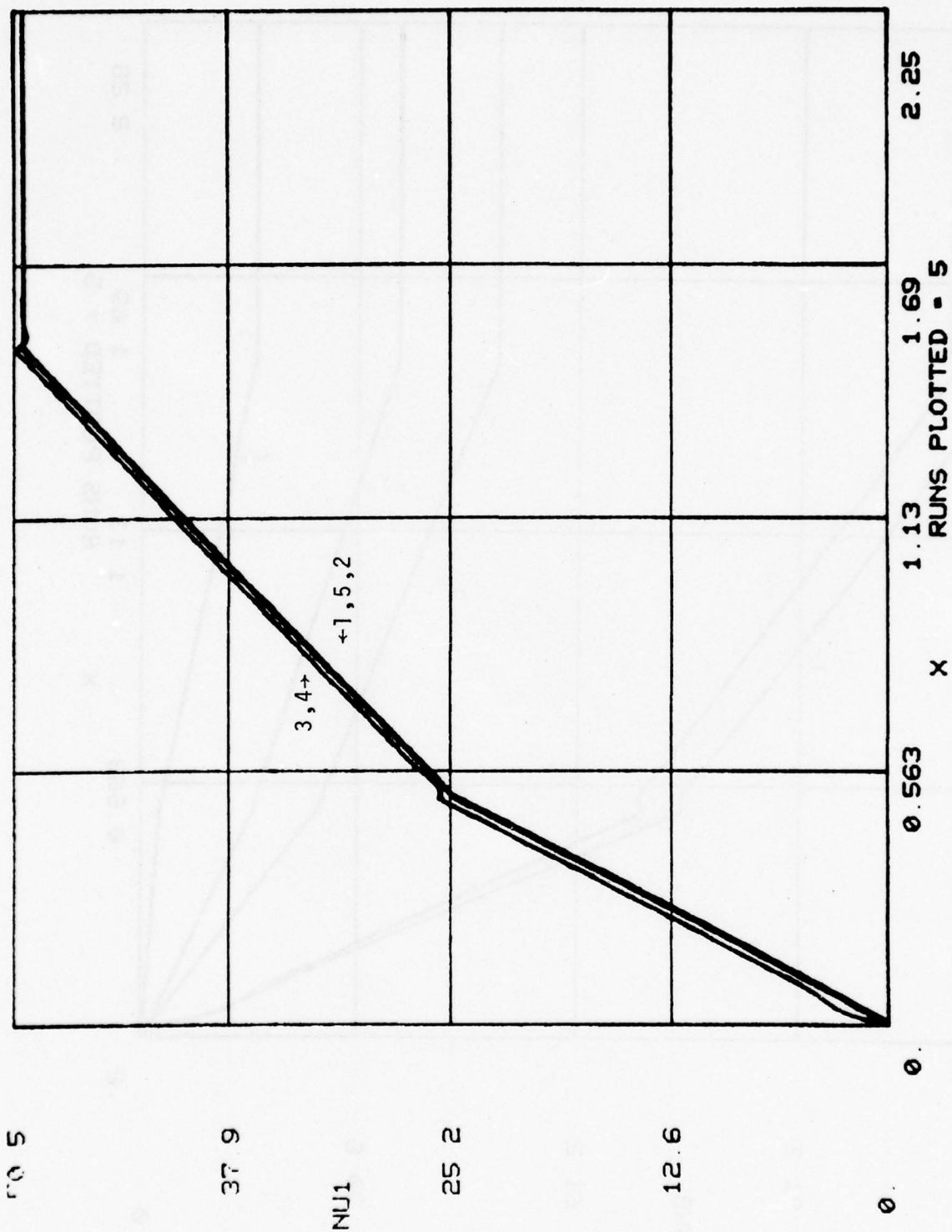


Figure 17

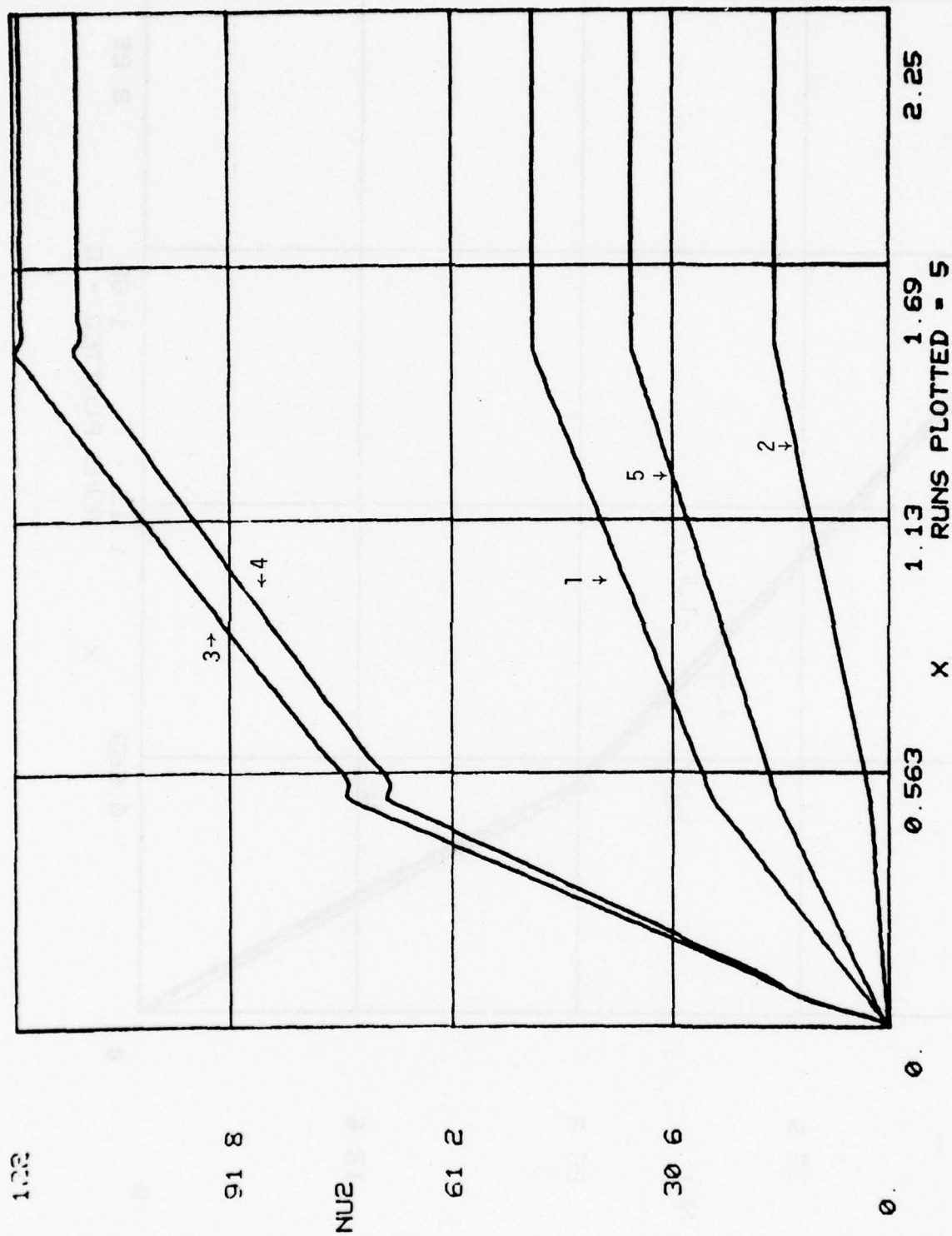


Figure 18

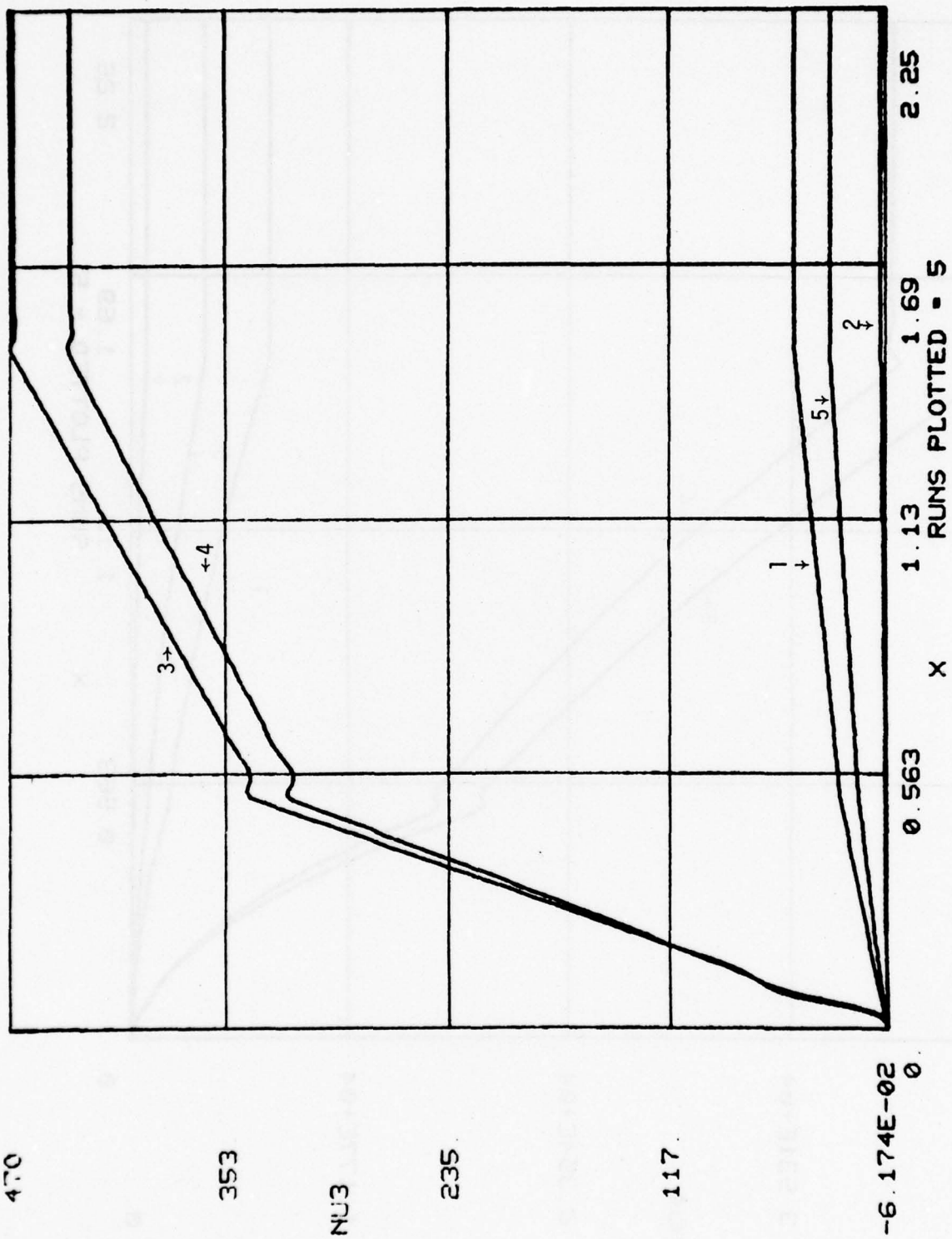
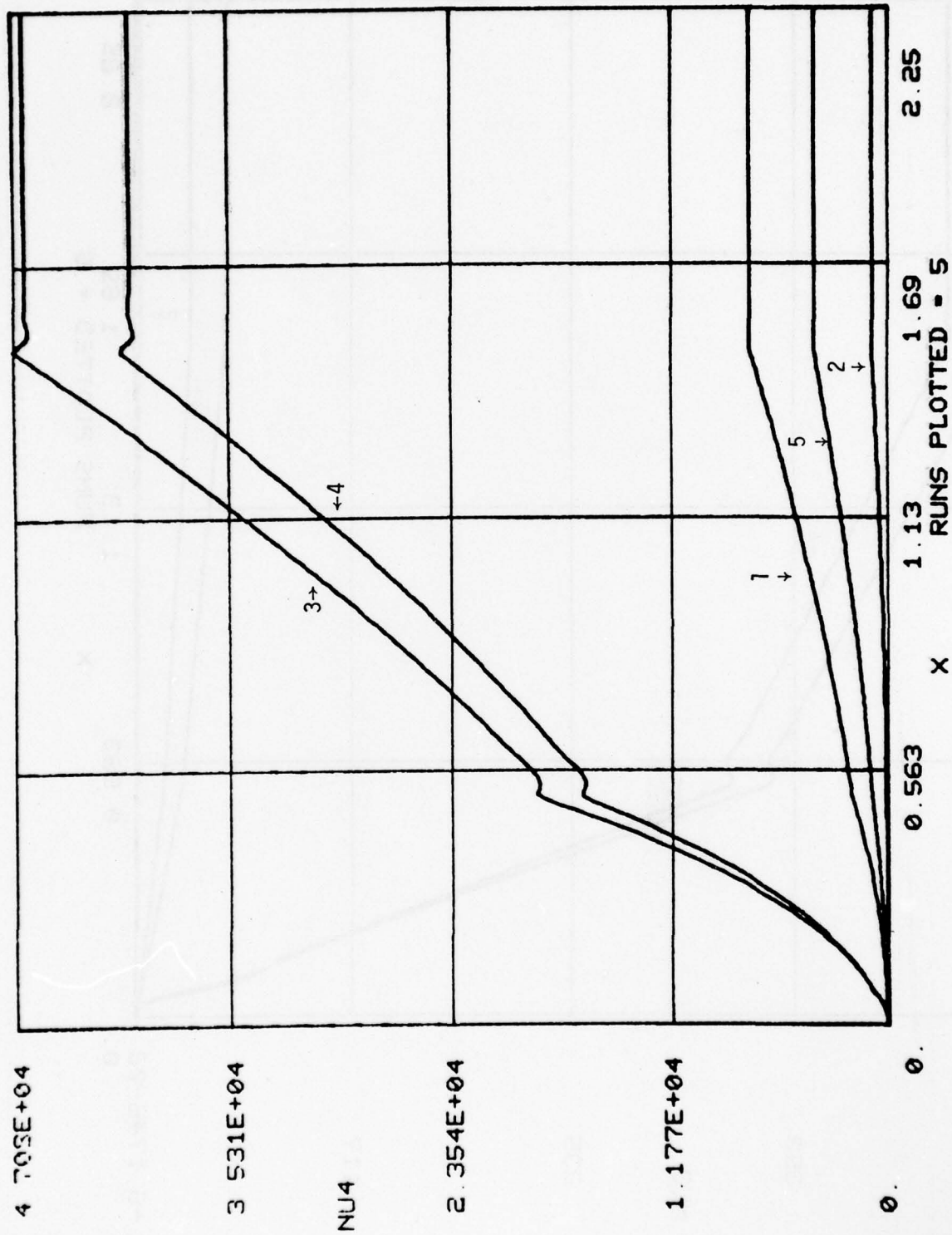


Figure 19



An examination of the graphs of the mean queue length (NU1) given in Figures 8, 12 and 16 shows that (for fixed service time distribution) each of the five PH-distributions considered here results in almost the same value for the mean queue length at every point x . The insensitivity of even the time-dependent mean queue length to substantial random variability in the arrival process may be deceptive. An examination of the computed curves for the time-dependent curves of the second, third and fourth central moments show that the latter are all highly sensitive to variability in the inter-arrival times. In fact, increased variability in the latter manifests itself in the same qualitative order in all the higher moments.

Although, for the initial conditions chosen in our examples, the approach to "steady-state" is very rapid, the higher values of the central moments can only be explained by a more erratic behavior of the path functions. These observations indicate that sole reliance on simple analytic expressions for mean queue lengths (where these are available) may lead to serious errors in the qualitative interpretation of the behavior of stochastic models.

CHAPTER IV

THE PH/PH/ ∞ QUEUE

4.1 Introduction

Consider the PH/G/ ∞ queue and assume that the service time distribution $\tilde{H}(\cdot)$ is also of phase type. Such a model will be denoted by PH/PH/ ∞ . In this Chapter we shall set up the differential equations for this model indicating their proofs briefly. In the course of our discussion we shall also present an alternate proof of an interesting theorem on the M/PH/ ∞ queue due to M. F. Neuts and J. L. Jain [8].

To be specific let us assume that the service time c.d.f. $\tilde{H}(\cdot)$ which is of Phase Type has a representation $(\underline{\beta}, S)$ and consists of n phases. Once again we assume that $\beta_{n+1}=0$ and without loss of generality that the representation $(\underline{\beta}, S)$ is so chosen that the PH-Renewal Process defined by it is irreducible.

For the PH/PH/ ∞ queue we are interested in the random variables $X_j(t)$ =the number of customers in phase j of service at time $t+$, $j=1, \dots, n$ and $J(t)$ =the phase of the arrival process at time $t+$. We let, for $k_i \geq 0$, $t \geq 0$, $1 \leq i, j \leq n$,

$$G_{ij}(k_1, \dots, k_n, t) = P \left[J(t)=j, X_1(t)=k_1, \dots, X_n(t)=k_n \mid J(0)=i, \right. \\ \left. X_1(0)=\dots=X_n(0)=0 \right] \quad (4.1.1)$$

and denote the $m \times m$ matrix defined by these entries by $G(k_1, \dots, k_n, t)$. For $t \geq 0$ and $|z_i| \leq 1$, $i=1, \dots, n$ we also define the generating function

$$G(z_1, \dots, z_n, t) = \sum_{k_1 \geq 0, \dots, k_n \geq 0} z_1^{k_1} z_2^{k_2} \dots z_n^{k_n} G(k_1, \dots, k_n, t). \quad (4.1.2)$$

In the sequel we shall simply write \underline{k} and \underline{z} respectively to denote the vectors (k_1, \dots, k_n) and (z_1, \dots, z_n) , and it will be implicitly understood that $k_i \geq 0$ is an integer for $i=1, \dots, n$ and $|z_i| \leq 1$ for all $i=1, \dots, n$. We also let

$$S_r(t) = \{(u_1, \dots, u_r) : u_i \geq 0, \sum u_i \leq t\}, \quad t \geq 0, \quad r \geq 1.$$

Further for $\underline{u} \in S_r(t)$, $g_r(t, \underline{u}, \underline{k})$ will denote the probability of obtaining E_1 k_1 times, ..., E_n k_n times in r multinomial trials each of which can result in any one of the $n+1$ mutually exclusive and collectively exhaustive events E_1, \dots, E_{n+1} with the trials having the probabilities given by the vectors

$$(\underline{g} \exp[S(t - u_1 - \dots - u_i)], 1 - \underline{g} \exp[S(t - u_1 - \dots - u_i)] \cdot \underline{e}); \quad i=1, \dots, r$$

respectively. Note that $\underline{g}_r(t, \underline{u}, \underline{k})$, where $\underline{k} \leq r$, is the conditional probability, given there are r arrivals in $(0, t]$ and these occur at $u_1, u_1 + u_2, \dots, u_1 + \dots + u_r$, that k_j of these customers are in phase j of service at time $t+$ for $j=1, \dots, n$ and the rest $r - \sum_{j=1}^n k_j$ depart in $(0, t]$. We also have

$$\begin{aligned}
g_r(t, \underline{u}, \underline{z}) &\stackrel{\text{def}}{=} \sum_{\substack{k \geq 0 \\ \underline{k} \leq r}} z_1^{k_1} \dots z_n^{k_n} g_r(t, \underline{u}, \underline{k}) \\
&= \prod_{i=1}^r \left\{ \beta \exp[S(t - u_1 - \dots - u_i)] \cdot \underline{z} + 1 - \beta \exp[S(t - u_1 - \dots - u_i)] \cdot \underline{e} \right\}
\end{aligned} \tag{4.1.3}$$

With these notations we are now ready to discuss the PH/PH/ ∞ queue.

4.2 The Analysis of the PH/PH/ ∞ Queue

Lemma 4.2.1: For $t \geq 0$,

$$\begin{aligned}
G(\underline{0}, t) &= \exp(Tt) + \sum_{r=1}^{\infty} \int_{S_r(t)} \left\{ \prod_{i=1}^r \exp(Tu_i) \cdot T^{\circ} A^{\circ} \right\} \cdot \\
&\quad \exp[T(t - u_1 - \dots - u_r)] g_r(t, \underline{u}, \underline{0}) d\underline{u}
\end{aligned}$$

and for $\underline{k} > \underline{0}$,

$$\begin{aligned}
G(\underline{k}, t) &= \sum_{r=\underline{k}}^{\infty} \int_{S_r(t)} \left\{ \prod_{i=1}^r \exp(Tu_i) \cdot T^{\circ} A^{\circ} \right\} \cdot \exp[T(t - u_1 - \dots - u_r)] \\
&\quad g_r(t, \underline{u}, \underline{k}) d\underline{u}
\end{aligned}$$

Proof: This Lemma is obtained by conditioning on r , the number of arrivals in $(0, t]$, and the arrival epochs $u_1, u_1 + u_2, \dots, u_1 + \dots + u_r$, and applying the law of total probability.

Theorem 4.2.2: (Basic System of Differential Equations):

For $t \geq 0$,

$$\frac{\partial}{\partial t} G(\underline{z}, t) = [(T + T^{\circ} A^{\circ}) + \{\beta \exp(St) \cdot (\underline{z} - \underline{e})\} T^{\circ} A^{\circ}] G(\underline{z}, t) \tag{4.2.3}$$

and

$$G(\underline{z}, 0) = I \tag{4.2.4}$$

Proof: The above Theorem is obtained by using Lemma 4.2.1, (4.1.2) and (4.1.3). We omit the details which are analogous to those in the proof of Theorem 2.2.2.

By applying Proposition (2.2.8) to (4.2.3), (4.2.4) we have

Theorem 4.2.5: The basic system of differential equations for the PH/PH/ ∞ queue given by (4.2.3), (4.2.4) has a unique solution in $[0, \infty)$. Further this solution is bounded in $[0, \infty)$.

Using Levinson's Theorem quoted as Proposition 2.4.1, we have

Theorem 4.2.6: As $t \rightarrow \infty$, the solution $G(\underline{z}, t)$ to (4.2.3), (4.2.4) converges to a matrix $G(\underline{z})$ all whose rows are identical. Also $G(\underline{e}') = \theta$ where $\underline{e}' = (1, \dots, 1) \in \mathbb{R}^n$.

This implies

Theorem 4.2.7: The process $\{(X_1(t), \dots, X_n(t), J(t)) : t \geq 0\}$ is ergodic and has a stationary distribution independent of the initial conditions. This stationary distribution is given by $G(\underline{z})$.

Before we proceed with the discussion of the moments, we present the following interesting result on the M/PH/ ∞ queue due to M. F. Neuts and J. L. Jain [8].

Theorem 4.2.8: Consider the M/PH/ ∞ queue (which is obtained by setting $\alpha=1$ and $T=-\lambda$ in the PH/PH/ ∞ model). Given $X_1(0)=\dots=X_n(0)=0$, $X_1(t), \dots, X_n(t)$ are independent for any fixed $t>0$, and for $j=1, \dots, n$, $X_j(t)$ has a Poisson distribution with parameter $\lambda \int_0^t \underline{\beta} \exp(S\tau) \underline{e}_j d\tau$, where \underline{e}_j is the n -vector whose j -th component is one and all other components are zero. Also the process $(X_1(t), \dots, X_n(t))$ is ergodic, and in the steady-state these r.v.s. are independent with X_j following a Poisson distribution with parameter $-\lambda \underline{\beta} S^{-1} \underline{e}_j$.

Proof: For the M/PH/ ∞ queue we get from (4.2.3) and (4.2.4),

$$\begin{aligned} \frac{\partial}{\partial t} G(\underline{z}, t) &= \lambda \{ \underline{\beta} \exp(S t) \cdot (\underline{z} - \underline{e}) \} G(\underline{z}, t) \\ G(\underline{z}, 0) &= 1. \end{aligned}$$

It is easily verified that the solution to the above is given by

$$G(\underline{z}, t) = \prod_{j=1}^n \exp \left[-\lambda \int_0^t \underline{\beta} \exp(S\tau) \cdot \underline{e}_j d\tau \cdot (1 - z_j) \right]$$

whence the theorem.

Remark: Note that $-\underline{\beta} S^{-1} \underline{e}_j = \int_0^{\infty} \underline{\beta} \exp(S\tau) \underline{e}_j d\tau$ is the expected time spent by a customer in phase j of service. In the light of this fact, the values obtained for the parameters of the Poisson distributions in the above theorem are very intuitive.

We now return to the general discussion of the PH/PH/ ∞ queue and define the factorial moments

$$\mu_{ij}^{(k)} = E(X_j^{(k)}(t) | J(0)=i, X_1(0)=\dots=X_n(0)=0), \quad k \geq 1, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

Also the vector

$$\underline{\mu}_j^{(k)}(t) = (\mu_{1j}^{(k)}(t), \dots, \mu_{mj}^{(k)}(t))', \quad j=1, \dots, n, \quad k \geq 1.$$

By differentiating (4.2.3) with respect to z_j and setting $\underline{z}=\underline{e}$, we can obtain

Theorem 4.2.9: For $t \geq 0$, $1 \leq j \leq n$,

$$\left. \begin{aligned} \frac{d}{dt} \underline{\mu}_j^{(1)}(t) &= (T + T \circ A \circ) \underline{\mu}_j^{(1)}(t) + \{\underline{\beta} e^{St} \underline{e}_j\} T \circ \\ \underline{\mu}_j^{(1)}(0) &= \underline{0} \end{aligned} \right\} \quad (4.2.10)$$

and for $k \geq 2$

$$\left. \begin{aligned} \frac{d}{dt} \underline{\mu}_j^{(k)}(t) &= (T + T \circ A \circ) \underline{\mu}_j^{(k)}(t) + k \{\underline{\beta} e^{St} \underline{e}_j\} T \circ A \circ \underline{\mu}_j^{(k-1)}(t) \\ \underline{\mu}_j^{(k)}(0) &= \underline{0}. \end{aligned} \right\} \quad (4.2.11)$$

Further as $t \rightarrow \infty$, for any $k \geq 1$, $1 \leq j \leq n$, $\underline{\mu}_j^{(k)}(t)$ converges to a finite vector all whose components are equal.

We conclude our discussion by presenting the following

Corollary 4.2.10: For $1 \leq j \leq n$, as $t \rightarrow \infty$, $\underline{\mu}_j^{(1)}(t)$ converges to a vector each of whose components is equal to $-\lambda \underline{\beta} S^{-1} \underline{e}_j$, where λ^{-1} is the mean inter-arrival time given by $\lambda^{-1} = -\underline{\alpha} T^{-1} \underline{e}$.

Remark: Noting that $-\underline{\beta} S^{-1} \underline{e}_j$ is the expected time spent by a customer in phase j of service, we see that the above result is very intuitive.

Proof: From (4.2.10),

$$\underline{\theta} \underline{\mu}_j^{(1)}(t) = \underline{\theta} \underline{I}^0 \int_0^t (\underline{\beta} e^{S\tau} \underline{e}_j) d\tau$$

$$\rightarrow \underline{\theta} \underline{I}^0 (-\underline{\beta} S^{-1} \underline{e}_j) \text{ as } t \rightarrow \infty..$$

$$= -\lambda \underline{\beta} S^{-1} \underline{e}_j,$$

for, $\underline{\theta} = -\lambda \underline{\alpha} \underline{I}^{-1}$ and $\underline{I}^0 = -\underline{I} \underline{e}$.

The proof is complete by appealing to the last statement of Theorem 4.2.9.

ACKNOWLEDGEMENT

We wish to thank Professor D. E. Lamb, Department of
Statistics & Computer Science, University of Delaware,
Newark, Delaware for the use of the DELSIM software system.

BIBLIOGRAPHY

BIBLIOGRAPHY

- [1] Bellman (1960)
Introduction to Matrix Analysis
McGraw-Hill Book Company, Inc., New York
- [2] Cesari (1963)
Asymptotic Behavior and Stability Problems in Ordinary Differential Equations
Springer Verlag
- [3] Levinson (1948)
The Asymptotic Nature of Solutions of Linear Systems of Differential Equations
Duke Math. Jr., Vol. 15, pp. 111-126.
- [4] Loève (1963)
Probability Theory
Van Nostrand, 3rd Edition
- [5] Neuts, M. F. (1975)
Probability Distributions of Phase Type
in--Liber Amicorum Professor Emeritus H. Florin--
Department of Mathematics, University of Louvain,
Belgium, pp. 173-206.
- [6] Neuts, M. F. (1976)
Renewal Processes of Phase Type
Mimeo. Ser. #76/8, Department of Statistics & Computer
Science, University of Delaware, Newark, DE 19711, USA;
forthcoming in Nav. Res. Logistics Quarterly
- [7] Neuts, M. F. (1976)
Markov Chains with Applications in Queueing Theory
which have a Matrix-geometric Invariant Probability
Vector
Mimeo. Ser. #76/7, Department of Statistics & Computer
Science, University of Delaware, Newark, DE 19711
forthcoming in Adv. Appl. Prob.
- [8] Neuts, M. F. and Jain, J. L. (1977)
Unpublished Personal Communication

- [9] Royden, H. L. (1963)
Real Analysis
The Macmillan Company, New York
- [10] Takács, L. (1962)
Introduction to the Theory of Queues
Oxford University Press, New York

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 78-0981	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SOME EXPLICIT FORMULAS AND COMPUTATIONAL METHODS FOR INFINITE SERVER QUEUES WITH PHASE TYPE ARRIVALS		5. TYPE OF REPORT & PERIOD COVERED Interim
7. AUTHOR(s) V. Ramaswami and Marcel F. Neuts		6. PERFORMING ORG REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of Delaware Dept. of Statistics & Computer Science Newark, DE 19711		8. CONTRACT OR GRANT NUMBER(s) AFOSR-77-3236
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling AFB, Washington, DC 20332		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304/A5
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		12. REPORT DATE April 1978
		13. NUMBER OF PAGES 62
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This paper discusses infinite server queues whose input is a Phase Type Renewal Process, introduced by M.F. Neuts (Renewal Processes of Phase Type, Nav. Res. Log. Quart, forthcoming). The problems of obtaining the transient and steady-state distributions and moments of the queue length are reduced to the solution of certain well-behaved systems of linear differential equations. Sample computations are provided with as many as ten phases. The paper contains some useful		

DD FORM 1 JAN 73 1473

EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

next page

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

20.

explicit formulas and also discusses the interesting special case where the service time is also of phase type. The Phase Type Distributions include a wide variety of models such as generalized Erlang, hyperexponential (mixtures of a finite number of exponentials) as very special cases and possess great versatility in modeling a number of interesting qualitative features such as bimodality.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)